Supplementary Book of First Course College Mathematics

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Chapter One

1. Propositional Logic and Set Theory

In this chapter, we study the basic concepts of propositional logic and some part of set theory. In the first part, we deal about propositional logic, logical connectives, quantifiers and arguments. In the second part, we turn our attention to set theory and discus about description of sets and operations of sets.

1.1 Propositional Logic

Mathematical or symbolic logic is an analytical theory of the art of reasoning whose goal is to systematize and codify principles of valid reasoning. It has emerged from a study of the use of language in argument and persuasion and is based on the identification and examination of those parts of language which are essential for these purposes. It is formal in the sense that it lacks reference to meaning. Thereby it achieves versatility: it may be used to judge the correctness of a chain of reasoning (in particular, a "mathematical proof") solely on the basis of the form (and not the content) of the sequence of statements which make up the chain. There is a variety of symbolic logics. We shall be concerned only with that one which encompasses most of the deductions of the sort encountered in mathematics. Within the context of logic itself, this is "classical" symbolic logic.

Definition and examples of propositions

Consider the following sentences.

a. 2 is an even number.
b. A triangle has four sides.
c. Emperor Menelik ate chicken soup the night after the battle of Adwa.
d. May God bless you!
f. What is your name?
e. Give me that book.

The first three sentences are declarative sentences. The first one is true and the second one is false. The truth value of the third sentence cannot be ascertained because of lack of historical records but it is, by its very form, either true or false but not both. On the other hand, the last three sentences have not truth value. So they are not declaratives.

Now we begin by examining proposition, the building blocks of every argument. A proposition is a sentence that may be asserted or denied. Proposition in this way are different from questions, commands, and exclamations. Neither questions, which can be asked, nor exclamations, which can be uttered, can possibly be asserted or denied. Only propositions assert that something is (or is not) the case, and therefore only they can be true or false.

Definition 1.1: A proposition (or statement) is a sentence which has a truth value (either True or False but not both).

The above definition does not mean that we must always know what the truth value is. For example, the sentence "The 1000^{th} digit in the decimal expansion of π is 7" is a proposition, but it may be necessary to find this information in a Web site on the Internet to determine whether this statement is true. Indeed, for a sentence to be a proposition (or a statement), it is not a requirement that we be able to determine its truth value.

Remark: Every proposition has a truth value, namely **true** (denoted by ***T***) or **false** (denoted by ***F***).

1.1.1 Logical connectives

In mathematical discourse and elsewhere one constantly encounters declarative sentences which have been formed by modifying a sentence with the word "not" or by connecting sentences with the words "and", "or", "if . . . then (or implies)", and "if and only if". These five words or combinations of words are called propositional connectives.

Note: Letters such as p, q, r, s etc. are usually used to denote actual propositions.

Conjunction

When two propositions are joined with the connective "**and**," the proposition formed is a logical ***conjunction***. "and" is denoted by "$\wedge$". So, the logical conjunction of two propositions, p and q, is written:

$p \wedge q$, read as "$p$and q," or "p conjunction q".

p and q are called ***the components of the conjunction***.$p \wedge q$is true if and only if p is true and q is true.

The truth table for conjunction is given as follows:

p	q	$p \wedge q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***F***

Example 1.1: Consider the following propositions:

p: 3 is an odd number. (True)

q: 27 is a prime number. (False)

r: Addis Ababa is the capital city of Ethiopia. (True)

a. $p \wedge q$: 3 is an odd number and 27 is a prime number. (False)
b. $p \wedge r$: 3 is an odd number and Addis Ababa is the capital city of Ethiopia. (True)

Disjunction

When two propositions are joined with the connective "**or**," the proposition formed is called a logical ***disjunction***. "or" is denoted by "$\vee$". So, the logical disjunction of two propositions, p and q, is written:

$p \vee q$ read as "p or q" or "p disjunction q."

$p \vee q$ isfalse if and only if both p and q are false.

The truth table for disjunction is given as follows:

p	q	$p \vee q$
T	***T***	***T***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Example 1.2: Consider the following propositions:

p: 3 is an odd number. (True)

q: 27 is a prime number. (False)

s: Nairobi is the capital city of Ethiopia. (False)

a. $p \vee q$: 3 is an odd number or 27 is a prime number. (True)
b. p ∨ s: 27 is a prime number orNairobi is the capital city of Ethiopia. (False)

Note: The use of "**or**" in propositional logic is rather different from its normal use in the English language. For example, if Solomon says, "I will go to the football match in the afternoon or I will go to the cinema in the afternoon," he means he will do one thing or the other, but not both.

Here "or" is used in the exclusive sense. But in propositional logic, "or" is used in the inclusive sense; that is, we allow Solomon the possibility of doing both things without him being inconsistent.

Implication

When two propositions are joined with the connective "**implies**," the proposition formed is called a *logical* ***implication***. "implies" is denoted by "$\Longrightarrow$." So, the logical implication of two propositions, p and q, is written:

$p \Longrightarrow q$read as "p implies q."

The function of the connective "implies" between two propositions is the same as the use of "If … then …" Thus $p \Longrightarrow q$ can be read as "if p, then q."

$p \Longrightarrow q$is false if and only if p is true and q is false.

This form of a proposition is common in mathematics. The proposition p is called the hypothesis or the antecedent of the conditional proposition $p \Longrightarrow q$ while q is called its conclusion or the consequent.

The following is the truth table for implication.

p	q	$p \Longrightarrow q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***T***
F	***F***	***T***

Examples 1.3: Consider the following propositions:

p: 3 is an odd number. (True)

q: 27 is a prime number. (False)

r: Addis Ababa is the capital city of Ethiopia. (True)

$p \Longrightarrow q$: If 3 is an odd number, then 27 is prime. (False)

$p \Longrightarrow r$: If 3 is an odd number, then Addis Ababa is the capital city of Ethiopia. (True)

We have already mentioned that the implication $p \Longrightarrow q$ can be expressed as both "If p, then q" and "p implies q." There are various ways of expressing the proposition $p \Longrightarrow q$, namely:

If p, then q.

$q$if p.

$p$implies q.

$p$only if q.

$p$is sufficient for q.

$q$is necessary for p

Bi-implication

When two propositions are joined with the connective "**bi-implication**," the proposition formed is called a *logical**bi-implication*** or a *logical **equivalence***. A bi-implication is denoted by "$\Leftrightarrow$". So the logical bi-implication of two propositions, p and q, is written:

$p \Leftrightarrow q$.$p \Leftrightarrow q$is false if and only if p and $q$have different truth values.

The truth table for bi-implication is given by:

p	q	$p \Leftrightarrow q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***T***

Examples 1.4:

a. Let p: 2 is greater than 3. (False)
q: 5 is greater than 4. (True)
Then
$p \Leftrightarrow q$: 2 is greater than 3 if and only if 5 is greater than 4. (False)
b. Consider the following propositions:
p: 3 is an odd number. (True)
q: 2 is a prime number. (True)

$p \Leftrightarrow q$: 3 is an odd number if and only if 2 is a prime number. (True)

There are various ways of stating the proposition $p \Leftrightarrow q$.

$p$if and only if q (also written as p iff q),

$p$implies$q$ and q implies p,

$p$is necessary and sufficient for q

$q$is necessary and sufficient for p

$p$is equivalent to q

Negation

Given any proposition p, we can form the proposition $\neg p$ called the **negation** of p. The truth value of $\neg p$ is F if p is T and T if $p$is F.

We can describe the relation between p and $\neg p$ as follows.

p	$\neg p$
T	**F**
F	**T**

Example 1.5: Let p: Addis Ababa is the capital city of Ethiopia. (True)

$\neg p$: Addis Ababa is not the capital city of Ethiopia. (False)

Exercises

1. Which of the following sentences are propositions? For those that are, indicate the truth value.
   a. 123 is a prime number.
   b. 0 is an even number.
   c. $x^2 - 4 = 0$.
   d. Multiply $5x + 2$ by 3.
   e. What an impossible question!
2. State the negation of each of the following statements.
   a. $\sqrt{2}$is a rational number.
   b. 0 is not a negative integer.
   c. 111 is a prime number.
3. Let p: 15 is an odd number.
   q: 21 is a prime number.
   State each of the following in words, and determine the truth value of each.

a. $p \vee q$.
b. $p \wedge q$.
c. $\neg p \vee q$.
d. $p \wedge \neg q$.
e. $p \Longrightarrow q$.
f. $q \Longrightarrow p$.
g. $\neg q \Longrightarrow \neg p$.

1.1.2 Compound (or complex) propositions

So far, what we have done is simply to define the logical connectives, and express them through algebraic symbols. Now we shall learn how to form propositions involving more than one connective, and how to determine the truth values of such propositions.

Definition 1.2: The proposition formed by joining two or more proposition by connective(s) is called a compound statement.

Note: We must be careful to insert the brackets in proper places, just as we do in arithmetic. For example, the expression$p \Longrightarrow q \wedge r$will be meaningless unless we know whichconnective should apply first.It could mean$(p \Longrightarrow q) \wedge r$or $p \Longrightarrow (q \wedge r)$, which arevery different propositions.The truth value of such complicated propositions is determined by systematic applications of the rules for the connectives.

The possible truth values of a proposition are often listed in a table, called a **truth table**. If p and q are propositions, then there are four possible combinations of truth values for $p$and q. That is,TT, TF, FT and FF. If a third proposition r is involved, then there are eight possible combinations of truth values for p,q and r. In general, a truth table involving "n" propositions p_1,p_2,…,p_n contains 2^n possible combinations of truth values for these propositions and a truth table showing these combinations would have n columns and 2^n rows. So, we use truth tables to determine the truth value of a compound proposition based on the truth value of its constituent component propositions.

Examples1.6:

a. Suppose p and r are true and q and s are false.

What is the truth value of$(p \wedge q) \Longrightarrow (r \vee s)$?

i. Since p is true and q is false, $p \wedge q$ is false.
ii. Since r is true and s is false, $r \vee s$ is true.
iii. Thus by applying the rule of implication, we get that $(p \wedge q) \Longrightarrow (r \vee s)$ istrue.

b. Suppose that a compound proposition is symbolized by

$$(p \vee q) \Longrightarrow (r \Leftrightarrow \neg s)$$

and that the truth values of$p, q, r,$ and s are $T, F, F,$ and T, respectively. Then the truth value of $p \vee q$is T, that of $\neg s$ is **F**, that of $r \Leftrightarrow \neg s$ is T. So the truth value of $(p \vee q) \Longrightarrow (r \Leftrightarrow \neg s)isT$.

Remark: When dealing with compound propositions, we shall adopt the following convention on the use of parenthesis. Whenever "$\vee$" or "$\wedge$" occur with "$\Longrightarrow$" or "$\Longleftrightarrow$", we shall assume that "$\vee$" or "$\wedge$" is applied first, and then "$\Longrightarrow$" or "$\Longleftrightarrow$" is then applied. For example,

$p \wedge q \Longrightarrow r$ means $(p \wedge q) \Longrightarrow r$

$p \vee q \Longleftrightarrow r$ means $(p \vee q) \Longleftrightarrow r$

$\neg q \Longrightarrow \neg p$ means $(\neg q) \Longrightarrow (\neg p)$

$\neg q \Longrightarrow p \Longleftrightarrow r$ means $((\neg q) \Longrightarrow p) \Longleftrightarrow r$

However, it is always advisable to use brackets to indicate the order of the desired operations.

Definition 1.3: Two compound propositions P and Q are said to be ***equivalent*** if they have the same truth value for all possible combinations of truth values for the component propositions occurring in both P and Q. In this case we write $P \equiv Q$.

Example 1.7: Let $P: p \Longrightarrow q$.

$Q: \neg q \Longrightarrow \neg p$.

p	q	$\neg p$	$\neg q$	$p \Longrightarrow q$	$\neg q \Longrightarrow \neg p$
T	***T***	***F***	***F***	***T***	***T***
T	***F***	***F***	***T***	***F***	***F***
F	***T***	***T***	***F***	***T***	***T***
F	***F***	***T***	***T***	***T***	***T***

Then, P is equivalent to Q, since columns 5 and 6 of the above table are identical.

Example 1.8: Let $P: p \Longrightarrow q$.

$Q: \neg p \Longrightarrow \neg q$.

Then

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$
T	***T***	***F***	***F***	***T***	***T***
T	***F***	***F***	***T***	***F***	***T***
F	***T***	***T***	***F***	***T***	***F***
F	***F***	***T***	***T***	***T***	***T***

Looking at columns 5 and 6 of the table we see that they are not identical. Thus $P \not\equiv Q$.

It is useful at this point to mention the non-equivalence of certain conditional propositions. Given the conditional $p \Rightarrow q$, we give the related conditional propositions:-

$q \Rightarrow p$: Converse of $p \Rightarrow q$

$\neg p \Rightarrow \neg q$: Inverse of $p \Rightarrow q$

$\neg q \Rightarrow \neg p$: Contrapositive of $p \Rightarrow q$

As we observed from example 1.7, the conditional $p \Rightarrow q$ and its contrapositve $\neg q \Rightarrow \neg p$ are equivalent. On the other hand, $p \Rightarrow q \not\equiv q \Rightarrow p$and $p \Rightarrow q \not\equiv \neg p \Rightarrow \neg q$.

Do not confuse the contra positive and the converse of the conditional proposition. Here is the difference:

Converse: The hypothesis of a converse statement is the conclusion of the conditional statement and the conclusion of the converse statement is the hypothesis of the conditional statement.

Contra positive: The hypothesis of a contra positive statement is the negation of conclusion of the conditional statement and the conclusion of the contra positive statement is the negation of hypothesis of the conditional statement.

Example 1.9:

a. If Kidist lives in Addis Ababa, then she lives in Ethiopia.
   Converse: If Kidist lives in Ethiopia, then she lives in Addis Ababa.
   Contrapositive: If Kidist does not live in Ethiopia, then she does not live in Addis Ababa.
   Inverse: If Kidist does not live in Addis Ababa, then she does not live in Ethiopia.
b. If it is morning, then the sun is in the east.
   Converse: If the sun is in the east, then it is morning.

Contrapositive: If the sun is not in the east, then it is not morning.
Inverse: If it is not morning, then the sun is not the east.

Propositions, under the relation of logical equivalence, satisfy various laws or identities, which are listed below.

1. Idempotent Laws
   a. $p \equiv p \vee p$.
   b. $p \equiv p \wedge p$.
2. Commutative Laws
   a. $p \wedge q \equiv q \wedge p$.
   b. $p \vee q \equiv q \vee p$.
3. Associative Laws
   a. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.
   b. $p \vee (q \vee r) \equiv (p \vee q) \vee r$.
4. Distributive Laws
   a. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
   b. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
5. De Morgan's Laws
   a. $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
   b. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
6. Law of Contrapositive
   $p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$
7. Complement Law
   $\neg(\neg p) \equiv p$.

1.1.3Tautology and contradiction

Definition: A compound proposition is a **tautology** if it is always true regardless of the truth values of its component propositions. If, on the other hand, a compound proposition is always false regardless of its component propositions, we say that such a proposition is a **contradiction**.

Examples 1.10:

a. Suppose p is any proposition. Consider the compound propositions $p \vee \neg p$and $p \wedge \neg p$.

$\boldsymbol{p}$	$\neg p$	$\boldsymbol{p} \vee \neg p$	$\boldsymbol{p} \wedge \neg p$
T	**F**	***T***	***F***
F	**T**	***T***	***F***

Observe that $p \vee \neg p$ is a tautology while $p \wedge \neg p$ is a contradiction.

b. For any propositions $p$and q. Consider the compound proposition$p \Longrightarrow (q \Longrightarrow p)$. Let us make a truth table and study the situation.

p	q	$q \Rightarrow p$	$p \Rightarrow (q \Rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

We have exhibited all the possibilities and we see that for all truth values of the constituent propositions, the proposition $p \Rightarrow (q \Rightarrow p)$ is always true. Thus, $p \Rightarrow (q \Rightarrow p)$ is a tautology.

c. The truth table for the compound proposition $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$.

p	q	$\neg q$	$p \wedge \neg q$	$p \Rightarrow q$	$(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

In example 1.10(c), the given compound proposition has a truth value F for every possible combination of assignments of truth values for the component propositions $p$and q. Thus $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$ is a contradiction.

Remark:

1. In a truth table, if a proposition is a tautology, then every line in its column has$T$ as its entry;if a proposition is a contradiction, every line in its column has$F$ as its entry.
2. Two compound propositions P and Q are equivalent if and only if "$P \Leftrightarrow Q$" is a tautology.

Exercises

1. For statements $p, q$and r, use a truth table to show that each of the following pairs of statements is logically equivalent.
   a. $(p \wedge q) \Leftrightarrow p$and $p \Rightarrow q$.
   b. $p \Rightarrow (q \vee r)$and $\neg q \Rightarrow (\neg p \vee r)$.
   c. $(p \vee q) \Rightarrow r$and $(p \Rightarrow q) \wedge (q \Rightarrow r)$.

d. $p \Longrightarrow (q \vee r)$and $(\neg r) \Longrightarrow (p \Longrightarrow q)$.

e. $p \Longrightarrow (q \vee r)$and $((\neg r) \wedge p) \Longrightarrow q$.

2. For statements p, q, and r, show that the following compound statements are tautology.
   a. $p \Longrightarrow (p \vee q)$.
   b. $(p \wedge (p \Longrightarrow q)) \Longrightarrow q$.
   c. $\big((p \Rightarrow q) \wedge (q \Rightarrow r)\big) \Longrightarrow (p \Rightarrow r)$.
3. For statements $p$and q, show that $(p \wedge \neg q) \wedge (p \wedge q)$ is a contradiction.
4. Write the contra positive and the converse of the following conditional statements.
   a. If it is cold, then the lake is frozen.
   b. If Solomon is healthy, then he is happy.
   c. If it rains, Tigist does not take a walk.
5. Let p and q be statements. Which of the following implies that $p \vee q$ is false?
   a. $\neg p \vee \neg q$is false.
   b. $\neg p \vee q$is true.
   c. $\neg p \wedge \neg q$is true
   d. $p \Longrightarrow q$is true.
   e. $p \wedge q$is false.
6. Suppose that the statements p, q, r, and s are assigned the truth values T, F, F,and T, respectively. Find the truth value of each of the following statements.
   a. $(p \vee q) \vee r$.
   b. $p \vee (q \vee r)$.
   c. $r \Longrightarrow (s \wedge p)$.
   d. $p \Longrightarrow (r \Longrightarrow s)$.
   e. $p \Longrightarrow (r \vee s)$.
   f. $(p \vee r) \Leftrightarrow (r \wedge \neg s)$.
   g. $(s \Leftrightarrow p) \Longrightarrow (\neg p \vee s)$.
   h. $(q \wedge \neg s) \Longrightarrow (p \Leftrightarrow s)$.
   i. $(r \wedge s) \Longrightarrow (p \Longrightarrow (\neg q \vee s))$.
   j. $(p \vee \neg q) \vee r \Longrightarrow (s \wedge \neg s)$.
7. Suppose the value of $p \Longrightarrow q$is T; what can be said about the value of $\neg p \wedge q \Leftrightarrow p \vee q$?
8. a. Suppose the value of $p \Leftrightarrow q$ is T; what can be said about the values of $p \Leftrightarrow \neg q$ and $\neg p \Leftrightarrow q$?

   b. Suppose the value of $p \Leftrightarrow q$ is F; what can be said about the values of $p \Leftrightarrow \neg q$ and $\neg p \Leftrightarrow q$?
9. Construct the truth table for each of the following statements.
   a. $p \Longrightarrow (p \Longrightarrow q)$.
   b. $(p \vee q) \Leftrightarrow (q \vee p)$.
   c. $p \Longrightarrow \neg(q \wedge r)$.
   d. $(p \Longrightarrow q) \Leftrightarrow (\neg p \vee q)$.
   e. $\big(\mathrm{p} \Longrightarrow (\mathrm{q} \wedge \mathrm{r})\big) \vee (\neg \mathrm{p} \wedge \mathrm{q})$.
   f. $(p \wedge q) \Longrightarrow ((q \wedge \neg q) \Longrightarrow (r \wedge q))$.
10. For each of the following determine whether the information given is sufficient to decide the truth value of the statement. If the information is enough, state the truth value. If it is insufficient, show that both truth values are possible.
    a. $(p \Longrightarrow q) \Longrightarrow r$, where $r = T$.
    b. $p \wedge (q \Longrightarrow r)$, where $q \Longrightarrow r = T$.
    c. $p \vee (q \Longrightarrow r)$, where $q \Longrightarrow r = T$.

d. $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$, where $p \vee q = T$.
e. $(p \Longrightarrow q) \Longrightarrow (\neg q \Longrightarrow \neg p)$, where $q = T$.
f. $(p \wedge q) \Longrightarrow (p \vee s)$, where $p = T$and $s = F$.

1.1.4 Open propositions and quantifiers

In mathematics, one frequently comes across sentences that involve a variable. For example,$x^2 + 2x - 3 = 0$ is one such.The truth value of this statement depends on the value we assign for the variable x. For example, if $x = 1$, then this sentence is true, whereas if $x = -1$, then the sentence is false.

Definition 1.4: An open statement (also called a predicate) is a sentence that contains one or more variables and whose truth value depends on the values assigned for the variables. We represent an open statement by a capital letter followed by the variable(s) in parenthesis, e.g., $P(x), Q(x)$ etc.

Example 1.11: Here are some open propositions:

a. $x$is the day before Sunday.
b. $y$is a city in Africa.
c. $x$is greater than y.
d. $x + 4 = -9$.

It is clear that each one of these examples involves variables, but is not a proposition as we cannot assign a truth value to it. However, if individuals are substituted for the variables, then each one of them is a proposition or statement. For example, we may have the following.

a. Monday is the day before Sunday.
b. London is a city in Africa.
c. 5 is greater than 9.
d. –13 + 4= –9

Remark

The collection of all allowable values for the variable in an open sentence is called the **universal set**(the universe of discourse) and denoted by $\boldsymbol{U}$.

Definition 1.5: Two open proposition $P(x)$ and $Q(x)$ are said to be equivalent if and only if

$P(a) = Q(a)$for all individual a. Note that if the universe U is specified, then $P(x)$ and $Q(x)$ are equivalent if and only if $P(a) = Q(a)$ for all $a \in U$.

Example 1.12: Let $P(x): x^2 - 1 = 0$.

$Q(x): |x| \geq 1$.

Let $U = \{-1, -\frac{1}{2}, 0,1\}$.

Then for all $a \in U$; $P(a)$ and $Q(a)$ have the same truth value.

$P(-1): (-1)^2 - 1 = 0$	(T)	$Q(-1): \lvert -1 \rvert \geq 1$	(T)
$P\left(-\frac{1}{2}\right): (-\frac{1}{2})^2 - 1 = 0$	(F)	$Q\left(-\frac{1}{2}\right): \left\lvert -\frac{1}{2} \right\rvert \geq 1$	(F)
$P(0): 0 - 1 = 0$	(F)	$Q(0): \lvert 0 \rvert \geq 1$	(F)
$P(1): 1 - 1 = 0$	(T)	$Q(1): \lvert 1 \rvert \geq 1$	(T)

Therefore $P(a) = Q(a)$ for all $a \in U$.

Definition 1.6: Let U be the universal set. An open proposition $P(x)$ is a tautology if and only if $P(a)$ is always true for all values of $a \in U$.

Example 1.13: The open proposition $P(x): x^2 \geq 0$ is a tautology.

As we have observed in example 1.11, an open proposition can be converted into a proposition by substituting the individuals for the variables. However, there are other ways that an open proposition can be converted into a proposition, namely by a method called quantification. Let $P(x)$ be an open proposition over the domain S. Adding the phrase "For every $x \in S$" to $P(x)$ or "For some $x \in S$" to $P(x)$ produces a statement called a quantified statement.

Consider the following open propositions with universe $\mathbb{R}$.

a. $R(x): x^2 \geq 0$.
b. $P(x): (x + 2)(x - 3) = 0$.
c. $Q(x): x^2 < 0$.

Then $R(x)$ is always true for each $x \in \mathbb{R}$,

$P(x)$is true only for $x = -2$ and $x = 3$,

$Q(x)$is always false for all values of $x \in \mathbb{R}$.

Hence, given an open proposition $P(x)$, with universe U, we observe that there are three possibilities.

a. $P(x)$is true for all $x \in U$.
b. $P(x)$is true for some $x \in U$.
c. $P(x)$is false for all $x \in U$.

Now we proceed to study open propositions which are satisfied by "**all**" and "**some**" members of the given universe.

a. The phrase "for every x " is called a **universal quantifier**. We regard "for every x," "for all x," and "for each x " as having the same meaning and symbolize each by "$(\forall x)$." Think of the symbol $\forall$ as an inverted A(representing all). If $P(x)$ is an open proposition with universe$U$, then $(\forall x)P(x)$ is a quantified proposition and is read as "every $x \in U$ has the property P."

b. The phrase "there exists an x " is called an **existential quantifier**. We regard "there exists an$x$," "for some x," and "for at least one x " as having the same meaning, and symbolize each by "$(\exists x)$." Think of the symbol $\exists$ as the backwards capital E(representing exists). If $P(x)$ is an open proposition with universe U, then $(\exists x)P(x)$ is a quantified proposition and is read as "there exists $x \in U$ with the property P."

Remarks:

i. To show that$(\forall x)P(x)$is F, it is sufficient to find at least one$a \in U$ such that $P(a)$ is F. Such an element $a \in U$ is called a ***counter example***.

ii. $(\exists x)P(x)$is$F$ if we cannot find any $a \in U$ having the property P.

Example 1.14:

a. Write the following statements using quantifiers.

  i. For each real number $x > 0, x^2 + x - 6 = 0$.

  Solution: $(\forall x > 0)(x^2 + x - 6 = 0)$.

  ii. There is a real number$x > 0$ such that $x^2 + x - 6 = 0$.

  Solution: $(\exists x > 0)(x^2 + x - 6 = 0)$.

  iii. The square of any real number is nonnegative.

  Solution: $(\forall x \in \mathbb{R})(x^2 \geq 0)$.

b.

  i. Let $P(x): x^2 + 1 \geq 0$. The truth value for $(\forall x)P(x)$ [i.e $(\forall x)(x^2 + 1 \geq 0)$] is T.

  ii. Let $P(x): x < x^2$. The truth value for $(\forall x)(x < x^2)$is F. $x = \frac{1}{2}$is a counterexample since $\frac{1}{2} \in \mathbb{R}$ but $\frac{1}{2} < \frac{1}{4}$. On the other hand, $(\exists x)P(x)$ is true, since $-1 \in \mathbb{R}$ such that $-1 < 1$.

  iii. Let $P(x): |x| = -1$. The truth value for $(\exists x)P(x)$ is F since there is no real number whose absolute value is -1.

Relationship between the existential and universal quantifiers

If $P(x)$ is a formula in x, consider the following four statements.

a. $(\forall x)P(x)$.

b. $(\exists x)P(x)$.

c. $(\forall x)\neg P(x)$.

d. $(\exists x)\neg P(x)$.

We might translate these into words as follows.

a. Everything has property P.
b. Something has property P.
c. Nothing has property P.
d. Something does not have property P.

Now (d) is the denial of (a), and (c) is the denial of (b), on the basis of everyday meaning. Thus, for example, the existential quantifier may be defined in terms of the universal quantifier.

Now we proceed to discuss the negation of quantifiers. Let $P(x)$ be an open proposition. Then $(\forall x)P(x)$ is false only if we can find an individual "a" in the universe such that $P(a)$ is false. If we succeed in getting such an individual, then $(\exists x)\neg P(x)$ is true. Hence $(\forall x)P(x)$ will be false if $(\exists x)\neg P(x)$ is true. Therefore the negation of $(\forall x)P(x)$ is $(\exists x)\neg P(x)$. Hence we conclude that

$$\neg(\forall x)P(x) \equiv (\exists x)\neg P(x).$$

Similarly, we can easily verified that

$$\neg(\exists x)P(x) \equiv (\forall x)\neg P(x).$$

Remark: To negate a statement that involves the quantifiers $\forall$ and $\exists$, change each $\forall$ to $\exists$, change each $\exists$ to $\forall$, and negate the open statement.

Example 1.15:

Let $U = \mathbb{R}$.

a. $\neg(\exists x)(x < x^2) \equiv (\forall x)\neg(x < x^2)$
$\equiv (\forall x)(x \geq x^2)$.
b. $\neg(\forall x)(4x + 1 = 0) \equiv (\exists x)\neg(4x + 1 = 0)$
$\equiv (\exists x)(4x + 1 \neq 0)$.

Given propositions containing quantifiers we can form a compound proposition by joining them with connectives in the same way we form a compound proposition without quantifiers. For example, if we have $(\forall x)P(x)$ and $(\exists x)Q(x)$ we can form $(\forall x)P(x) \Leftrightarrow (\exists x)Q(x)$.

Consider the following statements involving quantifiers. Illustrations of these along with translations appear below.

a. All rationals are reals. $(\forall x)(\mathbb{Q}(x) \Longrightarrow \mathbb{R}(x))$.
b. No rationals are reals. $(\forall x)(\mathbb{Q}(x) \Longrightarrow \neg\mathbb{R}(x))$.
c. Some rationals are reals. $(\exists x)(\mathbb{Q}(x) \wedge \mathbb{R}(x))$.
d. Some rationals are not reals. $(\exists x)(\mathbb{Q}(x) \wedge \neg\mathbb{R}(x))$.

Example 1.16:

Let $U =$ The set of integers.

Let $P(x)$: x is a prime number.

$Q(x)$: x is an even number.

$R(x)$: x is an odd number.

Then

a. $(\exists x)[P(x) \Longrightarrow Q(x)]$is T; since there is an x, say 2, such that $P(2) \Longrightarrow Q(2)$ is T.
b. $(\forall x)[P(x) \Longrightarrow Q(x)]$is F. As a counterexample take 7. Then $P(7)$ is T and $Q(7)$is F. Hence $P(7) \Longrightarrow Q(7)$.
c. $(\forall x)[R(x) \wedge P(x)]$is F.
d. $(\forall x)[(R(x) \wedge P(x)) \Longrightarrow Q(x)]$is F.

Quantifiers Occurring in Combinations

So far, we have only considered cases in which universal and existential quantifiers appear simply. However, if we consider cases in which universal and existential quantifiers occur in combination, we are lead to essentially new logical structures. The following are the simplest forms of combinations:

1. $(\forall x)(\forall y)P(x,y)$
   "for all x and for all y the relation $P(x,y)$ holds";

2. $(\exists x)(\exists y)P(x,y)$
"there is an x and there is a y for which $P(x,y)$holds";

3. $(\forall x)(\exists y)P(x,y)$
   "for every x there is a y such that $P(x,y)$holds";
4. $(\exists x)(\forall y)P(x,y)$
   "there is an x which stands to every y in the relation $P(x,y)$."

Example 1.17:

Let U = The set of integers.

Let $P(x,y)$: $x + y = 5$.

a. $(\exists x)$ $(\exists y)$ $P(x,y)$means that there is an integer x and there is an integer y such that $x + y = 5$. This statement is true when $x = 4$and $y = 1$, since 4 + 1 = 5. Therefore, the statement $(\exists x)$ $(\exists y)$ $P(x,y)$ is always true for this universe. There are other choices of x and y for which it would be true, but the symbolic statement merely says that there is at least one choice for x and y which will make the statement true, and we have demonstrated one such choice.

b. $(\exists x)\ (\forall y)\ P(x, y)$means that there is an integer x_0 such that for every y, $x_0 + y = 5$. This is false since no fixed value of x_0 will make this true for all y in the universe; e.g. if $x_0 = 1$, then $1 + y = 5$ is false for some y.

c. $(\forall x)\ (\exists y)\ P(x, y)$ means that for every integer x, there is an integer y such that $x + y = 5$. Let $x = a$, then $y = 5 - a$ will always be an integer, so this is a true statement.

d. $(\forall x)\ (\forall y)\ P(x, y)$means that for every integer x and for every integer y, $x + y = 5$. This is false, for if $x = 2$and $y = 7$, we get $2 + 7 = 9 \neq 5$.

Example 1.18:

a. Consider the statement

For every two real numbers $x$and y,$x^2 + y^2 \geq 0$.

If we let

$$P(x, y): x^2 + y^2 \geq 0$$

where the domain of both x and y is $\mathbb{R}$, the statement can be expressed as

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})P(x, y) \text{or as } (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 + y^2 \geq 0).$$

Since $x^2 \geq 0$ and $y^2 \geq 0$ for all real numbers $x$and y, it follows that $x^2 + y^2 \geq 0$ and so $P(x, y)$ is true for all real numbers x and y. Thus the quantified statement is true.

b. Consider the open statement

$$P(x, y): |x - 1| + |y - 2| \leq 2$$

where the domain of the variable x is the set E of even integers and the domain of the variable y is the set O of odd integers. Then the quantified statement

$$(\exists x \in E)(\exists y \in O)P(x, y)$$

can be expressed in words as

There exist an even integer x and an odd integer y such that $|x - 1| + |y - 2| \leq 2$.

Since $P(2,3): 1 + 1 \leq 2$ is true, the quantified statement is true.

c. Consider the open statement

$$P(x, y): xy = 1$$

where the domain of both x and y is the set $\mathbb{Q}^+$ of positive rational numbers. Then the quantified statement

$$(\forall x \in \mathbb{Q}^+)(\exists y \in \mathbb{Q}^+)P(x, y)$$

can be expressed in words as

For every positive rational number x, there exists a positive rational number y such that $xy = 1$.

It turns out that the quantified statement is true. If we replace $\mathbb{Q}^+$by $\mathbb{R}$, then we have

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})P(x, y) .$$

Since $x = 0$ and for every real number$y, xy = 0 \neq 1$, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})P(x, y)$ is false.

d. Consider the open statement

$$P(x, y): xy \text{is odd}$$

where the domain of both x and y is the set $\mathbb{N}$ of natural numbers. Then the quantified statement

$$(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})P(x, y),$$

expressed in words, is

There exists a natural number x such that for every natural numbers y, xy is odd. The statement is false.

In general, from the meaning of the universal quantifier it follows that in an expression $(\forall x)(\forall y)P(x, y)$ the two universal quantifiers may be interchanged without altering the sense of the sentence. This also holds for the existential quantifies in an expression such as $(\exists x)(\exists y)P(x, y)$.

In the statement $(\forall x)(\exists y)P(x, y)$, the choice of $y$is allowed to depend on x - the y that works for one x need not work for another x.On the other hand, in the statement $(\exists y)(\forall x)P(x, y)$, the y must work for all x, i.e., y is independent of x.For example, the expression$(\forall x)(\exists y)(x < y)$, where x and y are variables referring to the domain of real numbers, constitutes a true proposition, namely, "For every number x, there is a number y, such that x is less that y," i.e., "given any number, there is a greater number." However, if the order of the symbol $(\forall x)$ and $(\exists y)$ is changed, in this case, we obtain:$(\exists y)(\forall x)(x < y)$, which is a false proposition, namely, "There is a number which is greater than every number." By transposing $(\forall x)$and $(\exists y)$, therefore, we get a different statement.

The logical situation here is:

$$(\exists y)(\forall x)P(x,y) \Longrightarrow (\forall x)(\exists y)P(x,y).$$

Finally, we conclude this section with the remark that there are no mechanical rules for translating sentences from English into the logical notation which has been introduced. In every case one must first decide on the meaning of the English sentence and then attempt to convey that same meaning in terms of predicates, quantifiers, and, possibly, individual constants.

Exercise 1.1

1. In each of the following, two open statements $P(x,y)$ and $Q(x,y)$ are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x,y) \Longrightarrow Q(x,y)$ for the given values of x and y.
   a. $P(x,y): x^2 - y^2 = 0$. and $Q(x,y): x = y$. $(x,y) \in \{(1,-1),(3,4),(5,5)\}$.
   b. $P(x,y): |x| = |y|$. and $Q(x,y): x = y$. $(x,y) \in \{(1,2),(2,-2),(6,6)\}$.
   c. $P(x,y): x^2 + y^2 = 1$. and $Q(x,y): x + y = 1$. $(x,y) \in \{(1,-1),(-3,4),(0,-1),(1,0)\}$.
2. Let O denote the set of odd integers and let $P(x): x^2 + 1$ is even, and $Q(x): x^2$ is even. be open statements over the domain O. State $(\forall x \in O)P(x)$ and $(\exists y \in O)Q(x)$ in words.
3. State the negation of the following quantified statements.
   a. For every rational number r, the number $\frac{1}{r}$ is rational.
   b. There exists a rational number r such that $r^2 = 2$.
4. Let $P(n): \frac{5n-6}{3}$ is an integer. be an open sentence over the domain $\mathbb{Z}$. Determine, with explanations, whether the following statements are true or false:
   a. $(\forall n \in \mathbb{Z})P(n)$.
   b. $(\exists n \in \mathbb{Z})P(n)$.
5. Determine the truth value of the following statements.
   a. $(\exists x \in \mathbb{R})(x^2 - x = 0)$.
   b. $(\forall x \in \mathbb{N})(x + 1 \geq 2)$.
   c. $(\forall x \in \mathbb{R})(\sqrt{x^2} = x)$.
   d. $(\exists x \in \mathbb{Q})(3x^2 - 27 = 0)$.
   e. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 8)$.
   f. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 + y^2 = 9)$.
   g. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 5)$.
   h. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 5)$
6. Consider the quantified statement

   For every $x \in A$ and $y \in A$, $xy - 2$ is prime, where the domain of the variables x and y is $A = \{3,5,11\}$.
   a. Express this quantified statement in symbols.
   b. Is the quantified statement in (a) true or false? Explain.
   c. Express the negation of the quantified statement in (a) in symbols.
   d. Is the negation of the quantified in (a) true or false? Explain.

7. Consider the open statement $P(x,y)\colon \frac{x}{y} < 1$. where the domain of x is $A = \{2,3,5\}$ and the domain of $y$is $B = \{2,4,6\}$.
   a. State the quantified statement $(\forall x \in A)(\exists y \in B)P(x,y)$ in words.
   b. Show quantified statement in (a) is true.
8. Consider the open statement $P(x,y)\colon x - y < 0$. where the domain of x is $A = \{3,5,8\}$ and the domain of $y$is $B = \{3,6,10\}$.
   a. State the quantified statement $(\exists y \in B)(\forall x \in A)P(x,y)$ in words.

Show quantified statement in (a) is true.

1.1.5 Argument and Validity

Definition 1.7: An argument (logical deduction) is an assertion that a given set of statements $p_1, p_2, p_3, \dots, p_n$, called **hypotheses** or ***premises***, yield another statement Q, called the ***conclusion***. Such a logical deduction is denoted by:

$$p_1, p_2, p_3, \dots, p_n \vdash Q \text{ or}$$

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_n} \\ Q \end{array}$$

k

Example 1.19: Consider the following argument:

If you study hard, then you will pass the exam.
You did not pass the exam.
Therefore, you did not study hard.
Let p: You study hard.
q: You will pass the exam.
The argument form can be written as:

$$\begin{array}{r} p \Rightarrow q \\ \underline{\neg q} \\ \neg p \end{array}$$

When is an argument form accepted to be correct? In normal usage, we use an argument in order to demonstrate that a certain conclusion follows from known premises. Therefore, we shall require that under any assignment of truth values to the statements appearing, if the premises became all true, then the conclusion must also become true. Hence, we state the following definition.

Definition 1.8: An argument form $p_1, p_2, p_3, \dots, p_n \vdash Q$ is said to be ***valid*** if Q is true whenever all the premises $p_1, p_2, p_3, \dots, p_n$ are true; otherwise it is ***invalid***.

Example 1.20: Investigate the validity of the following argument:

a. $p \Rightarrow q, \neg q \vdash \neg p$

b. $p \Rightarrow q, \neg q \Rightarrow r \vdash p$

c. If it rains, crops will be good. It did not rain. Therefore, crops were not good.

Solution: First we construct a truth table for the statements appearing in the argument forms.

a.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$
T	***T***	***F***	***F***	***T***
T	***F***	***F***	***T***	***F***
F	***T***	***T***	***F***	***T***
F	***F***	***T***	***T***	***T***

The premises $p \Rightarrow q$ and $\neg q$ are true simultaneously in row 4 only. Since in this case p is also true, the argument is valid.

b.

p	q	r	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow r$
T	***T***	***T***	***F***	***T***	***T***
T	***T***	***F***	***F***	***T***	***T***
T	***F***	***T***	***T***	***F***	***T***
T	***F***	***F***	***T***	***F***	***F***
F	***T***	***T***	***F***	***T***	***T***
F	***T***	***F***	***F***	***T***	***T***
F	***F***	***T***	***T***	***T***	***T***
F	***F***	***F***	***T***	***T***	***F***

The 1st, 2nd, 5th, 6th and 7th rows are those in which all the premises take value T. In the 5th, 6th and 7th rows however the conclusion takes value$F$. Hence, the argument form is invalid.

c. Let p: It rains.
q: Crops are good.

$\neg p$: It did not rain.

$\neg q$: Crops were not good.

The argument form is $p \Longrightarrow q, \neg p \vdash \neg q$

Now we can use truth table to test validity as follows:

p	q	$\neg p$	$\neg q$	$p \Longrightarrow q$
T	***T***	***F***	***F***	***T***
T	***F***	***F***	***T***	***F***
F	***T***	***T***	***F***	***T***
F	***F***	***T***	***T***	***T***

The premises $p \Longrightarrow q$ and $\neg p$ are true simultaneously in row 4 only. Since in this case $\neg q$ is also true, the argument is valid.

Remark:

1. What is important in validity is the form of the argument rather than the meaning or content of the statements involved.
2. The argument form $p_1, p_2, p_3, \dots, p_n \vdash Q$ is valid iff the statement $(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \Longrightarrow Q$is a tautology.

Formal proof of validity of an argument

Definition 1.9: A formal proof of a conclusion Q given hypotheses $p_1, p_2, p_3, \dots, p_n$ is a sequence of stapes, each of which applies some inference rule to hypotheses or previously proven statements (antecedent) to yield a new true statement (the consequent).

A formal proof of validity is given by writing on the premises and the statements which follows from them in a single column, and setting off in another column, to the right of each statement, its justification. It is convenient to list all the premises first.

Example 1.21: Show that $p \Longrightarrow \neg q, q \vdash \neg p$ is valid.

Solution:

1.	q is true	premise
2.	$p \Longrightarrow \neg q$	premise
3.	$q \Longrightarrow \neg p$	contrapositive of (2)
4.	¬p	Modes Ponens using (1) and (3)

Example 1.22: Show that the hypotheses

It is not sunny this afternoon and it is colder than yesterday.

If we go swimming, then it is sunny.

If we do not go swimming, then we will take a canoe trip.

If we take a canoe trip, then we will be home by sunset.

Lead to the conclusion:

We will be home by sunset.

Let p: It is sunny this afternoon.

q: It is colder than yesterday.

r: We go swimming.

s: We take a canoe trip.

t: We will be home by sunset.

Then

1. $\neg p \wedge q$ hypothesis
2. $\neg p$ simplification using (1)
3. $r \Longrightarrow p$ hypothesis
4. $\neg r$ Modus Tollens using (2) and (3)
5. $\neg r \Longrightarrow s$ hypothesis
6. s Modus Ponens using (4) and (5)
7. $s \Longrightarrow t$ hypothesis
8. t Modus Ponens using (6) and (7)

Exercises 1.1

1. Use the truth table method to show that the following argument forms are valid.
   i. $\neg p \Longrightarrow \neg q, q \vdash p.$
   ii. $p \Longrightarrow \neg p, p, r \Longrightarrow q \vdash \neg r.$
   iii. $p \Longrightarrow q, \neg r \Longrightarrow \neg q \vdash \neg r \Longrightarrow \neg p.$
   iv. $\neg r \vee \neg s, (\neg s \Longrightarrow p) \Longrightarrow r \vdash \neg p.$
   v. $p \Longrightarrow q, \neg p \Longrightarrow r, r \Longrightarrow s \vdash \neg q \Longrightarrow s.$
2. For the following argument given a, b and c below:
   i. Identify the premises.
   ii. Write argument forms.

iii. Check the validity.

a. If he studies medicine, he will get a good job. If he gets a good job, he will get a good wage. He did not get a good wage. Therefore, he did not study medicine.

b. If the team is late, then it cannot play the game. If the referee is here, then the team is can play the game. The team is late. Therefore, the referee is not here.

c. If the professor offers chocolate for an answer, you answer the professor's question. The professor offers chocolate for an answer. Therefore, you answer the professor's question

3. Give formal proof to show that the following argument forms are valid.

a. $\neg p \Longrightarrow \neg q, q \vdash p.$

b. $p \Longrightarrow \neg q, p, r \Longrightarrow q \vdash \neg r.$

c. $p \Longrightarrow q, \neg r \Longrightarrow \neg q \vdash \neg r \Longrightarrow \neg p.$

d. $\neg r \wedge \neg s, (\neg s \Longrightarrow p) \Longrightarrow r \vdash \neg p.$

e. $p \Longrightarrow, \neg p \Longrightarrow r, r \Longrightarrow s \vdash \neg q \Longrightarrow s.$

f. $\neg p \vee q, r \Longrightarrow p, r \vdash q.$

g. $\neg p \wedge \neg q, (q \vee r) \Longrightarrow p \vdash \neg r.$

h. $p \Longrightarrow (q \vee r), \neg r, p \vdash q.$

i. $\neg q \Longrightarrow \neg p, r \Longrightarrow p, \neg q \vdash r.$

4. Prove the following are valid arguments by giving formal proof.

a. If the rain does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore, the rain comes.

If the team is late, then it cannot play the game. If the referee is here then the team can play the game. The team is late. Therefore, the referee is not here.

1.2 Set theory

In this section, we study some part of set theory especially description of sets, Venn diagrams and operations of sets.

1.2.1 The concept of a set

The term set is an undefined term, just as a point and a line are undefined terms in geometry. However, the concept of a set permeates every aspect of mathematics. Set theory underlies the language and concepts of modern mathematics. The term set refers to a well-defined collection of objects that share a certain property or certain properties. The term "**well-defined**" here means that the set is described in such a way that one can decide whether or not a given object belongs in the set. If A is a set, then the objects of the collection A are called the elements or members of the set$A$. If x is an element of the set$A$, we write $x \in A$. If x is not an element of the set$A$, we write$x \notin A$.

As a convention, we use capital letters to denote the names of sets and lowercase letters for elements of a set.

Note that for each objects x and each set A, exactly one of $x \in A$ or $x \notin A$ but not both must be true.

1.2.2 Description of sets

Sets are described or characterized by one of the following four different ways.

1. **Verbal Method**

In this method, an ordinary English statement with minimum mathematical symbolization of the property of the elements is used to describe a set. Actually, the statement could be in any language.

Example 1.23:

a. The set of counting numbers less than ten.
b. The set of letters in the word "Addis Ababa."
c. The set of all countries in Africa.

2. **Roster/Complete Listing Method**

If the elements of a set can all be listed, we list them all between a pair of braces without repetition separating by commas, and without concern about the order of their appearance. Such a method of describing a set is called ***the roster/complete listing*** method.

Examples 1.24:

a. The set of vowels in English alphabet may also be described as $\{a, e, i, o, u\}$.
b. The set of positive factors of 24 is also described as $\{1, 2, 3, 4, 6, 8, 12, 24\}$.

Remark:

i. We agree on the convention that the order of writing the elements in the list is immaterial. As a result the sets $\{a, b, c\}, \{b, c, a\}$ and $\{c, a, b\}$ contain the same elements, namely a, b and c.
ii. The set $\{a, a, b, b, b\}$ contains just two distinct elements; namely a and b, hence it is the same set as $\{a, b\}$. We list distinct elements without repetition.

Example 1.25:

a. Let $A = \{a, b, \{c\}\}$. Elements of A are a, b and $\{c\}$.
Notice that c and $\{c\}$ are different objects. Here $\{c\} \in A$ but $c \notin A$.
b. Let $B = \{\{a\}\}$. The only element of B is $\{a\}$. But $a \notin B$.
c. Let $C = \{a, b, \{a, b\}, \{a, \{a\}\}\}$. Then C has four elements.

The readers are invited to write down all the elements of C.

3. **Partial Listing Method**

In many occasions, the number of elements of a set may be too large to list them all; and in other occasions there may not be an end to the list. In such cases we look for a common property of the elements and describe the set by partially listing the elements. More precisely, if the common property is simple that it can easily be identified from a list of the first few elements, then with in a pair of braces, we list these few elements followed (or preceded) by exactly three dotes and possibly by one last element. The following are such instances of describing sets by partial listing method.

Example 1.26:

a. The set of all counting numbers is $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.
b. The set of non-positive integers is $\{\dots, -4, -3, -2, -1, 0\}$.
c. The set of multiples of 5 is $\{\dots, -15, -10, -5, 0\ 5, 10, 15, \dots\}$.
d. The set of odd integers less than 100 is $\{\dots, -3, -1, 1, 3, 5, \dots\ 99\}$.

4. **Set-builder Method**

When all the elements satisfy a common property P, we express the situation as an open proposition $P(x)$ and describe the set using a method called the ***Set-builder Method*** as follows:

$$A = \{x \mid P(x)\}\ or\ A = \{x : P(x)\}$$

We read it as "A is equal to the set of all x's such that $P(x)$ is true." Here the bar "| " and the colon ":" mean "such that." Notice that the letter $x$is only a place holder and can be replaced throughout by other letters. So, for a property P, the set $\{x \mid P(x)\}, \{t \mid P(t)\}$ and $\{y \mid P(y)\}$ are all the same set.

Example 1.27: The following sets are described using the set-builder method.

a. $A = \{x \mid x \text{ is a vowel in the English alphabet}\}$.
b. $B = \{t \mid t \text{ is an even integer}\}$.
c. $C = \{n \mid n \text{ is a natural number and } 2n - 15 \text{ is negative}\}$.
d. $D = \{y \mid y^2 - y - 6 = 0\}$.
e. $E = \{x \mid x \text{ is an integer and } x - 1 < 0 \Rightarrow x^2 - 4 > 0\}$.

Exercise: Express each of the above by using either the complete or the partial listing method.

Definition 1.10: The set which has no element is called the empty (or null) set and is denoted by ϕor {}.

Example 1.28: The set of $x \in \mathbb{R}$ such that $x^2 + 1 = 0$ is an empty set.

Relationships between two sets

Definition 1.11: Set B is said to be a ***subset*** of set A (or is contained in A), denoted by $B \subseteq A$, if every element of B is an element of A, i.e.,

$$(\forall x)(x \in B \Longrightarrow x \in A).$$

It follows from the definition that set B is not a subset of set A if at least one element of B is not an element of A. i.e., $B \nsubseteq A \Leftrightarrow (\exists x)(x \in B \Longrightarrow x \notin A)$. In such cases we write $B \nsubseteq A$or $A \nsupseteq B$.

Remarks: For any set $A, \phi \subseteq A$and $A \subseteq A$.

Example 1.29:

a. If $A = \{a, b\}$, $B = \{a, b, c\}$and $C = \{a, b, d\}$, then $A \subseteq B$ and $A \subseteq C$. On the other hand, it is clear that: $B \nsubseteq A$, $B \nsubseteq C$ and $C \nsubseteq B$.
b. If $S = \{x \mid x \text{ is a multiple of } 6\}$and $T = \{x \mid x \text{ is even integer}\}$, then $S \subseteq T$ since every multiple of 6 is even. However, $2 \in T$while$2 \notin S$. Thus$T \nsubseteq S$.
c. If $A = \{a, \{b\}\}$, then $\{a\} \subseteq A$and$\{\{b\}\} \subseteq A$. On the otherhand, since$b \notin A$, $\{b\} \nsubseteq A$, and $\{\text{a}, b\} \nsubseteq A$.

Definition 1.12: Sets A and B are said to be ***equal*** if they contain exactly the same elements. In this case, we write $A = B$. That is,

$(\forall x)(x \in B \Leftrightarrow x \in A)$.

Example 1.30:

a. The sets $\{1, 2, 3\}, \{2, 1, 3\}, \{1, 3, 2\}$ are all equal.
b. $\{\text{x} \mid \text{x is a counting number}\} = \{\text{x} \mid \text{x is a positive integer}\}$

Definition 1.13: Set A is said to be a **proper subset** of set B if every element of A is also an element of B, but B has at least one element that is not in A. In this case, we write $A \subset B$. We also say B is aproper super set of A, and write $B \supset A$. It is clear that

$A \subset B \Leftrightarrow [(\forall x)(x \in A \Longrightarrow x \in B) \wedge (A \neq B)]$.

Remark: Some authors do not use the symbol⊆. Instead they use the symbol ⊂ for both subset and proper subset. In this material, we prefer to use the notations commonly used in high school mathematics, and we continue using ⊆ and ⊂ differently, namely for subset and proper subset, respectively.

Definition 1.14: Let A be a set. The power set of A, dented by $P(A)$, is the set whose elements are all subsets of A. That is,

$P(A) = \{B: B \subseteq A\}$.

Example 1.31: Let $A = \{x, y, z\}$. As noted before, ϕ and A are subset of A. Moreover, $\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}$ and $\{y, z\}$ are also subsets of A. Therefore,

$$P(A) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, A\}.$$

Frequently it is necessary to limit the topic of discussion to elements of a certain fixed set and regard all sets under consideration as a subset of this fixed set. We call this set the **universal set** or the **universe** and denoted by $\boldsymbol{U}$.

Exercises 1.2

1. Which of the following are sets?
   a. 1,2,3
   b. {1,2},3
   c. {{1},2},3
   d. {1,{2},3}
   e. {1,2,a,b}.
2. Which of the following sets can be described in complete listing, partial listing and/or set-builder methods? Describe each set by at least one of the three methods.
   a. The set of the first 10 letters in the English alphabet.
   b. The set of all countries in the world.
   c. The set of students of Addis Ababa University in the 2018/2019 academic year.
   d. The set of positive multiples of 5.
   e. The set of all horses with six legs.
3. Write each of the following sets by listing its elements within braces.
   a. $A = \{x \in \mathbb{Z}: -4 < x \leq 4\}$
   b. $B = \{x \in \mathbb{Z}: x^2 < 5\}$
   c. $C = \{x \in \mathbb{N}: x^3 < 5\}$
   d. $D = \{x \in \mathbb{R}: x^2 - x = 0\}$
   e. $E = \{x \in \mathbb{R}: x^2 + 1 = 0\}$.
4. Let A be the set of positive even integers less than 15. Findthe truth value of each of the following.
   a. $15 \in A$
   b. $-16 \in A$
   c. $\phi \in A$
   d. $12 \subset A$
   e. $\{2, 8, 14\} \in A$
   f. $\{2,3,4\} \subseteq A$
   g. $\{2,4\} \in A$
   h. $\phi \subset A$
   i. $\{246\} \subseteq A$
5. Find the truth value of each of the following and justify your conclusion.
   a. $\phi \subseteq \phi$
   b. $\{1,2\} \subseteq \{1,2\}$
   c. $\phi \in A$ for any set A
   d. $\{\phi\} \subseteq A$, for any set A
   e. $5, 7 \subseteq \{5, 6, 7, 8\}$
   f. $\phi \in \{\{\phi\}\}$
   g. For any set $A, A \subset A$
   h. $\{\phi\} = \phi$
6. For each of the following set, find its power set.
   a. $\{ab\}$
   b. $\{1, 1.5\}$

c. $\{a, b\}$

d. $\{a, 0.5, x\}$

7. How many subsets and proper subsets do the sets that contain exactly $1, 2, 3, 4, 8, 10$ and 20 elements have?
8. If n is a whole number, use your observation in Problems 6and 7 to discover a formula for the number of subsets of a set with **n** elements. How many of these are proper subsets of the set?
9. Is there a set A with exactly the following indicated property?
   a. Only one subset
   b. Only one proper subset
   c. Exactly 3 proper subsets
   d. Exactly 4 subsets
   e. Exactly 6 proper subsets
   f. Exactly 30 subsets
   g. Exactly 14 proper subsets
   h. Exactly 15 proper subsets
10. How many elements does A contain if it has:
    a. 64 subsets?
    b. 31 proper subsets?
    c. No proper subset?
    d. 255 proper subsets?
11. Find the truth value of each of the following.
    a. $\phi \in P(\phi)$
    b. $For\ any\ set\ A, \phi \subseteq P(A)$
    c. $For\ any\ set\ A, A \in P(A)$
    d. $For\ any\ set\ A, A \subset P(A)$.
12. For any three sets A, B and C, prove that:
    a. If $A \subseteq B$and $B \subseteq C$, then $A \subseteq C$.
    b. If $A \subset B$and $B \subset C$, then $A \subset C$.

1.2.3 Set Operations and Venn diagrams

Given two subsets A and B of a universal set U, new sets can beformed using A and B in many ways, such as taking common elements or non-common elements, and putting everything together. Such processes of forming new sets are called ***set operations***. In this section, three most important operations, namely union, intersection and complement are discussed.

Definition 1.15: The union of two sets $A$and B, denoted by $A \cup B$, is the set of all elements that are either in A or in B (or in both sets). That is,

$$A \cup B = \{x: (x \in A) \vee (x \in B)\}.$$

As easily seen the union operator "∪" in the theory of set is the counterpart of the logical operator "∨".

Definition 1.16: The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements that are in A and B. That is,

$$A \cap B = \{x: (x \in A) \wedge (x \in B)\}.$$

As suggested by definition 1.15, the intersection operator "$\cap$" in the theory of sets is the counterpart of the logical operator "$\wedge$".

Note: - Two sets A and B are said to be disjoint sets if$A \cap B = \phi$.

Example 1.32:

a. Let $A = \{0, 1, 3, 5, 6\}$and$B = \{1, 2, 3, 4, 6, 7\}$. Then,
$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7\}$and $A \cap B = \{1, 3, 6\}$.

b. Let A = The set of positive even integers, and
B = The set of positive multiples of 3. Then,

$A \cup B = \{x: x \text{ is a positive intger that is either even or a multiple of } 3\}$
$= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, \dots\}$
$A \cap B = \{x \mid x \text{ is a positive integer that is both even and multiple of } 3\}$
$= \{6, 12, 18, 24, \dots\}$
$= \{x \mid x \text{ is a positive multiple of } 6.\}$

Example 1.33: If = {1,3,5}, $B = \{1,2\}$, then $A - B = \{3,5\}$ and $B - A = \{2\}$.

Note: The above example shows that, in general, $A - B$ are $B - A$ disjoint.

Definition 1.18: Let A be a subset of a universal set U. The ***absolute complement*** (or simply ***complement***) of A, denoted by A' (orA^cor$\bar{A}$), is defined to be the set of all elements of U that are not in A. That is,

$A' = \{x: x \in U \wedge x \notin A\}$or $x \in A' \Leftrightarrow x \notin A \Leftrightarrow \neg(x \in A)$.

Notice that taking the absolute complement of A is the same as finding the relative complement of A with respect to the universal set U. That is,

$$A' = U - A.$$

Example 1.34:

a. If $U = \{0,1,2,3,4\}$, and if $A = \{3,4\}$, then $A' = \{3,4\}$.

b. Let $U = \{1, 2, 3, \dots, 12\}$
$A = \{x \mid x \; is \; a \; positive \; factor \; of \; 12\}$
and $B = \{x \mid x \; is \; an \; odd \; integer \; in \; U\}$.
Then, $A' = \{5, 7, 8, 9, 10, 11\}$, $B' = \{2, 4, 6, 8, 10, 12\}$,
$(A \cup B)' = (8, 10\}$, $A' \cup B' = \{2, 4, 5, 6, \dots, 12\}$,
$A' \cap B' = \{8, 10\}$, and $(A \backslash B)' = \{1, 3, 5, 7, 8, 9, 10, 11\}$.

c. Let $U = \{a, b, c, d, e, f, g, h\}, A = \{a, e, g, h\}$ and $B = \{b, c, e, f, h\}$. Then $A' = \{b, c, d, f\}, B' = \{a, d, g\}, B - A = \{b, c, f\}$, $A - B = \{a, g\}$, and $(A \cup B)' = \{d\}$.

Find $(A \cap B)'$, $A' \cap B'$, $A' \cup B'$. Which of these are equal?

Theorem 1.1: For any two sets A and B, each of the following holds.

1. $(A')' = A$.
2. $A' = U - A$.
3. $A - B = A \cap B'$.
4. $(A \cup B)' = A' \cap B'$.
5. $(A \cap B)' = A' \cup B'$.
6. $A \subseteq B \Leftrightarrow B' \subseteq A'$.

Now we define the symmetric difference of two sets.

Definition 1.17: The symmetric difference of two sets A and B, denoted by $A\Delta B$, is the set

$A\Delta B = (A - B) \cup (B - A)$.

Example 1.35: Let $U = \{1,2,3, \ldots ,10\}$ be the universal set, $A = \{2,4,6,8,9,10\}$ and $B = \{3,5,7,9\}$. Then $B - A = \{3,5,7\}$ and $A - B = \{2,4,6,8,10\}$. Thus $A\Delta B = \{2,3,4,5,6,7,8,10\}$.

Theorem 1.2: For any three sets A, B and C, each of the following holds.

a. $A \cup B = B \cup A$. ($\cup$ is commutative)
b. $A \cap B = B \cap A$. ($\cap$ is commutative)
c. $(A \cup B) \cup C = A \cup (B \cup C)$. ($\cup$ is associative)
d. $(A \cap B) \cap C = A \cap (B \cap C)$. ($\cap$ is associative)
e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. ($\cup$ is distributive over $\cap$)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ($\cap$ is distributive over $\cup$)

Let us prove property "e" formally.

$x \in A \cup (B \cap C) \Leftrightarrow (x \in A) \vee (x \in B \cap C)$ (definition of $\cup$)

$\Leftrightarrow x \in A \vee (x \in B \wedge x \in C)$ (definition of $\cap$)

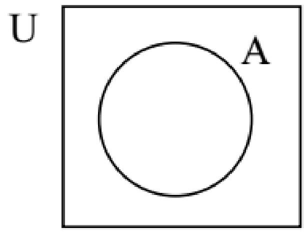
$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$ ($\vee$ is distributive over $\wedge$)
$\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C)$ (definition of $\cup$)
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$ (definition of $\cap$)

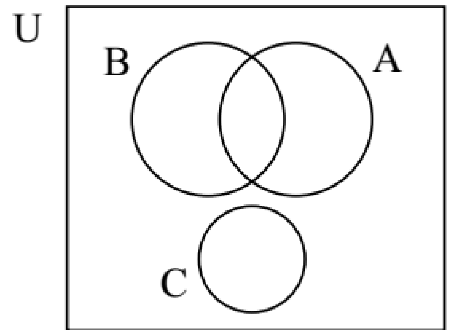
Therefore, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

The readers are invited to prove the rest part of theorem (1.2).

Venn diagrams

While working with sets, it is helpful to use diagrams, called **Venn diagrams,** to illustrate the relationships involved. A Venn diagram is a schematic or pictorial representative of the sets involved in the discussion. Usually sets are represented as interlocking circles, each of which is enclosed in a rectangle, which represents the universal set U.

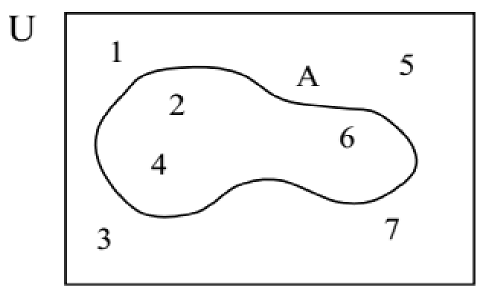




In some occasions, we list the elements of set A inside the closedcurve representing$A$.

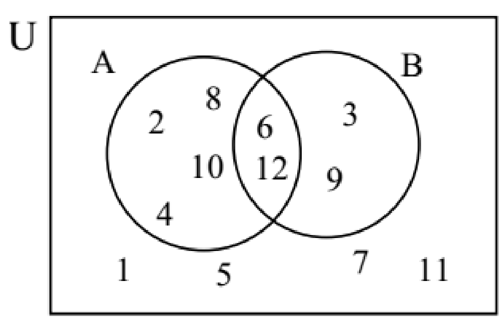
Example 1.36:

a. If $U = \{1, 2, 3, 4, 5, 6, 7\}$and $A = \{2, 4, 6\}$, then a Venndiagram representation of these two sets looks like thefollowing.



b. Let $U =$

$\{x \mid x \text{ is a positive integer less than } 13\}$
$A = \{x \mid x \in U \text{ and } x \text{ is even}\}$
$B = \{x \mid x \in U \text{ and } x \text{ is a multiple of } 3\}$.
A Venn diagram representation of these sets is given below.

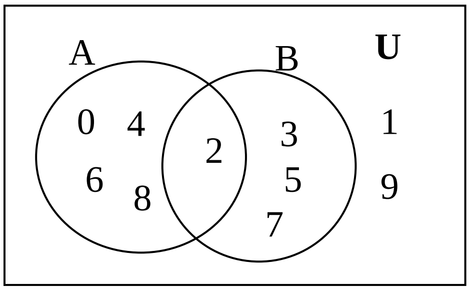


Example 1.37: Let U = The set of one digits numbers

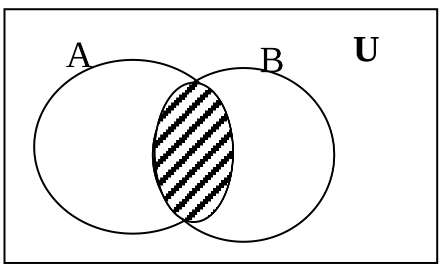
A = The set of one digits even numbers

B = The set of positive prime numbers less than 10

We illustrate the sets using a Venn diagram as follows.

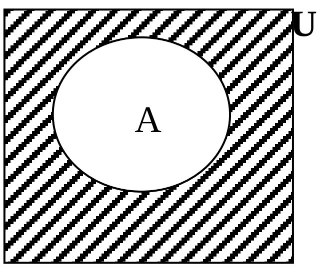


a. Illustrate $A \cap B$ by a Venn diagram



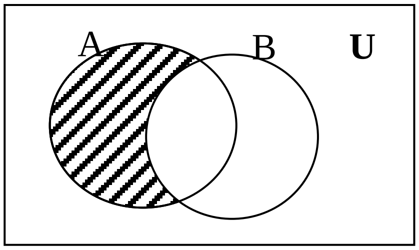
$A \cap B$: The shaded portion

b. Illustrate A' by a Venn diagram



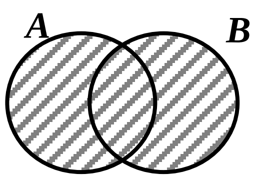
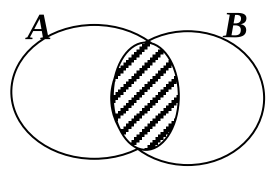
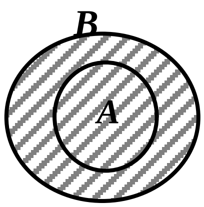
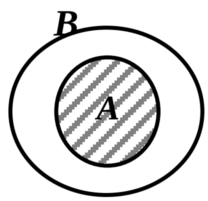
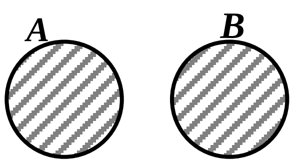
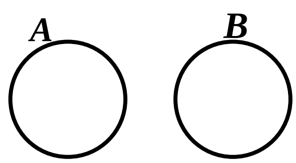
A':The shaded portion

c. Illustrate $A \backslash B$ by using a Venn diagram



$A \setminus B$: The shaded portion

Now we illustrate intersections and unions of sets by Venn diagram.

Cases	Shaded is $A \cup B$	Shaded $A \cap B$
Only some common elements		
$A \subseteq B$		
No common element		 $A \cap B = \varnothing$

Exercises 1.3

1. If $B \subseteq A$, $A \cap B' = \{1,4,5\}$ and $A \cup B = \{1,2,3,4,5,6\}$, find B.
2. Let $A = \{2,4,6,7,8,9\}$,
   $B = \{1,3,5,6,10\}$ and
   $C = \{ x: 3x + 6 = 0 \text{or } 2x + 6 = 0\}$. Find
   a. $A \cup B$.
   b. Is $(A \cup B) \cup C = A \cup (B \cup C)$?
3. Suppose $U =$ The set of one digit numbers and
   $A = \{x: x$ is an even natural number less than or equal to 9$\}$

Describe each of the sets by complete listing method:

a. A'.
b. $A \cap A'$.
c. $A \cup A'$.
d. $(A')'$
e. $\phi - U$.
f. ϕ'
g. U'

4. Suppose $U =$ The set of one digit numbers and
$A =\{x: x$ is an even natural number less than or equal to 9$\}$

Describe each of the sets by complete listing method:

a. A'
b. . $A \cap A'$.
c. $A \cup A'$.
d. $(A')'$
e. $\phi - U$.
f. ϕ'
g. U'

5. Use Venn diagram to illustrate the following statements:
   a. $(A \cup B)' = A' \cap B'$.
   b. $(A \cap B)' = A' \cup B'$.
   c. If $A \nsubseteq B$, then $A \backslash B \neq \phi$.
   d. $A \cup A' = U$.
6. Let $A = \{5,7,8,9\}$and $C = \{6,7,8\}$. Then show that $(A \backslash B) \backslash c = A(B \backslash C)$.
7. Perform each of the following operations.
   a. $\phi \cap \{\phi\}$
   b. $\{\phi, \{\phi\}\} - \{\{\phi\}\}$
   c. $\{\phi, \{\phi\}\} - \{\phi\}$
   d. $\{\{\{\phi\}\}\} - \phi$
8. Let $U = \{2, 3, 6, 8, 9, 11, 13, 15\}$,
$A = \{x | x$ is a positive prime factor of 66$\}$
$B =\{ x \in U | x$is composite number $\}$ and $C = \{x \in U |\ x - 5 \in U\}$. Then find each of the following.

$A \cap B, (A \cup B) \cap C, (A - B) \cup C, (A - B) - C, A - (B - C), (A - C) - (B - A), A' \cap B' \cap C'$

9. Let $A \cup B = \{a, b, c, d, e, x, y, z\}$and $A \cap B = \{b, e, y\}$.
   a. $If\ B - A = \{x, z\}$, then $A =$ ___________________
   b. $If\ A - B = \phi$, then $B =$ ___________________
   c. $If\ B = \{b, e, y, z\}$, then $A - B =$ _______________
10. Let $U = \{1, 2, \dots, 10\}, A = \{3, 5, 6, 8, 10\}, B = \{1, 2, 4, 5, 8, 9\}$,
$C = \{1, 2, 3, 4, 5, 6, 8\}$and $D = \{2, 3, 5, 7, 8, 9\}$. Verify each ofthe following.
    a. $(A \cup B) \cup C = A \cup (B \cup C)$.
    b. $A \cap (B \cup C \cup D) = (A \cap B) \cup (A \cap C) \cup (A \cap D)$.
    c. $(A \cap B \cap C \cap D)' = A' \cup B' \cup C' \cup D'$.
    d. $C - D = C \cap D'$.
    e. $A \cap (B \cap C)' = (A - B) \cup (A - C)$.
11. Depending on question No. 10find.
    a. $A \,\Delta\, B$.
    b. $C \,\Delta\, D$.

c. $(A \Delta C) \Delta D$.

d. $(A \cup B) \setminus (A \Delta B)$.

12. For any two subsets A and B of a universal set U, provethat:

a. $A \Delta B = B \Delta A$.

b. $A \Delta B = (A \cup B) - (A \cap B)$.

c. $A \Delta \phi = A$.

d. $A \Delta A = \phi$.

Chapter Two

The Real and the Complex Number System

In everyday life, knowingly or unknowingly, we are doing with numbers. Therefore, it will be nice if we get familiarized with numbers. Whatever course (which needs the concept of mathematics) we take, we face with the concept of numbers directly or indirectly. For this purpose, numbers and their basic properties will be introduced under this chapter.

2.1 The real number System

2.1.1 The set of natural numbers

The history of numbers indicated that the first set of numbers used by the ancient human beings for counting purpose was the set of natural (counting) numbers.

Definition 2.1.1

The set of natural numbers is denoted by **N** and is described as $\mathbf{N} = \{1, 2, 3, \ldots\}$

2.1.1.1 Operations on the set of natural numbers

i) Addition (+)

If two natural numbers *a* & *b* are added using the operation "+", then the sum *a*+*b* is also a natural number. If the sum of the two natural numbers *a* & *b* is denoted by *c*, then we can write the operation as: $c = a+b$, where *c* is called the sum and *a* & *b* are called terms.

Example: 3+8 = 11, here 11 is the sum whereas 3 & 8 are terms.

ii) Multiplication ($\times$)

If two natural numbers *a* & *b* are multiplied using the operation "$\times$", then the product $a \times b$ is also a natural number. If the product of the two natural numbers *a* & *b* is denoted by *c*, then we can write the operation as: $c = a \times b$, where *c* is called the product and *a* & *b* are called factors.

Example 2.1.3: $3 \times 4 = 12$, here 12 is the product whereas 3 & 4 are factors.

Properties of addition and multiplication on the set of natural numbers

i. For any two natural numbers *a* & *b*, the sum *a*+*b* is also a natural number. For instance in the above example, 3 and 8 are natural numbers, their sum 11 is also a natural number. In general, we say that the set of natural numbers is closed under addition.

ii. For any two natural numbers a & b, $a + b = b + a$.

Example 2.1.1: 3+8 = 8+3 = 11. In general, we say that addition is commutative on the set of natural numbers.

iii. For any three natural numbers a, b & c, $(a+b)+c = a+(b+c)$.

Example 2.1.2: (3+8)+6 = 3+(8+6) = 17. In general, we say that addition is associative on the set of natural numbers.

iv. For any two natural numbers a & b, the product $a \times b$ is also a natural number. For instance in the above example, 3 and 4 are natural numbers, their product 12 is also a natural number. In general, we say that the set of natural numbers is closed under multiplication.

v. For any two natural numbers a & b, $a \times b = b \times a$.

Example 2.1.4: $3 \times 4 = 4 \times 3 = 12$. In general, we say that multiplication is commutative on the set of natural numbers.

vi. For any three natural numbers a, b & c, $(a \times b) \times c = a \times (b \times c)$.

Example 2.1.5: $(2 \times 4) \times 5 = 2 \times (4 \times 5) = 40$. In general, we say that multiplication is associative on the set of natural numbers.

vi. For any natural number a, it holds that $a \times 1 = 1 \times a = a$.

Example 2.1.6: $6 \times 1 = 1 \times 6 = 6$. In general, we say that multiplication has an identity element on the set of natural numbers and 1 is the identity element.

vii. For any three natural numbers a, b & c, $a \times (b+c) = (a \times b)+(a \times c)$.

Example 2.1.7: $3 \times (5+7) = (3 \times 5)+ (3 \times 7) = 36$. In general, we say that multiplication is distributive over addition on the set of natural numbers.

Note: Consider two numbers a and b, we say a is greater than b denoted by $a > \text{b}$ if $a - b$ is positive.

2.1.1.2 Order Relation in N

i) **Transitive property**:

For any three natural numbers a, b & c, $a > b$ & $b > c \Rightarrow a > c$

ii) **Addition property:**

For any three natural numbers a, b & c, $a > b \Rightarrow a + c > b + c$

iii) **Multiplication property**:

For any three natural numbers a, b and c, $a > b \Rightarrow ac > bc$

iv) **Law of trichotomy**

For any two natural numbers a & b we have $a > b$ or $a < b$ or $a = b$.

Factors of a number

Definition 2.2

If a, b, c ϵ N such that ab=c, then a and b are factors (divisors) of c and c is called product (multiple) of a& b.

Example 2.8: Find the factors of 15.

Solution: Factors of 15 are 1, 3, 5, 15. *Or we can write it as*: $F_{15} = \{1, 3, 5, 15\}$

Definition 2.3 A number $a \in N$ is said to be

i. **Even** if it is divisible by 2.

ii. **Odd** if it is not divisible by 2.

iii. **Prime** if it has only two factors (1 and itself).

iv. **Composite**: if it has three or more factors

Example 2.9: 2, 4, 6, . . . are even numbers

Example 2.10: 1, 3, 5, . . . are odd numbers

Example 2.11: 2, 3, 5, . . . are prime numbers

Example 2.12: 4, 6, 8, 9, . . . are composite numbers

Remark: 1 is neither prime nor composite.

2.1.1.3 Prime Factorization

Definition 2.4

Prime factorization of a composite number is the product of all its prime factors.

Example 2.9:

$a)\ 6 = 2\times 3 \quad b)\ 30 = 2\times 3\times 5 \quad c)\ 12 = 2\times 2\times 3 = 2^2\times 3 \quad d)\ 8 = 2\times 2\times 2 = 2^3$

$e)\ 180 = 2^2\times 3^2\times 5$

Fundamental Theorem of Arithmetic:

Every composite number can be expressed as a product of its prime factors. This factorization is unique except the order of the factors.

2.1.1.4 Greatest Common Factor (GCF)

Definition 2.5

The greatest common factor (GCF) of two numbers *a* & *b* is denoted by GCF (*a*, *b*) and is the greatest number which is a factor of each of the given number.

Note: If the GCF of two numbers is 1, then the numbers are called relatively prime.

Example 2.10: Consider the two numbers 24 and 60.

$$\begin{aligned} &Now\ \ F_{24} = \{ 1, 2, 3, 4, 6, 8, 12, 24 \} \\ &and\ \ F_{60} = \{ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \} \\ &Next\ F_{24} \cap F_{60} = \{ 1, 2, 3, 4, 6, 12 \}\ from\ which\ 12\ is\ the\ greatest. \end{aligned}$$

Therefore, GCF(24, 60) = 12.

This method of finding the GCF of two or more numbers is usually lengthy and time consuming. Hence an alternative method (Prime factorization method) is provided as below:

Step 1: Find the prime factorization of each of the natural numbers

Step 2: Form the GCF of the given numbers as the product of every factor that appears in each of the prime factorization but take the least number of times it appears.

Example 2.11: Consider the two numbers 24 and 60.

$$\begin{aligned} Step1:\ 24 &= 2^3 \times 3 \\ 60 &= 2^2 \times 3 \times 5 \end{aligned}$$

Step 2: The factors that appear in both cases are 2 and 3, but take the numbers with the least number of times.

$$\therefore\ \ GCF(24, 60) = 2^2 \times 3 = \underline{\underline{12}}$$

Example 2.12: Consider the three numbers 20, 80 and 450.

$$\begin{aligned} Step1:\ 20 &= 2^2 \times 5 \\ 80 &= 2^4 \times 5 \\ 450 &= 2 \times 3^2 \times 5^2 \end{aligned}$$

Step 2: The factors that appear in all cases are 2 and 5, but take the numbers with the least number of times.

$\therefore \quad GCF\,(20,\ 80,\ 450) = 2 \times 5 = \underline{\underline{10}}$

2.1.1.5 Least Common Multiple (LCM)

Definition 2.6

The least common multiple (LCM) of two numbers *a* & *b* is denoted by LCM (*a*, *b*) and is the least number which is a multiple of each of the given number.

Example 2.13: Consider the two numbers 18 and 24.

$$\begin{aligned} &Now\ \ M_{18} = \{\ 18,\ 36,\ 54,\ 72, 90,\ 108,\ 126,\ 144,\ \cdots\ \} \\ &and\ \ M_{24} = \{\ 24,\ 48,\ 72,\ 96, 120,\ 144,\ \cdots\ \} \\ &Next\ \ M_{18} \cap M_{24} = \{\ 72,\ \ 144,\ \cdots\ \}\ \ from\ which\ \ 72\ is\ the\ least. \end{aligned}$$

Therefore, LCM (18, 24) = 72.

This method of finding the LCM of two or more numbers is usually lengthy and time consuming. Hence an alternative method (Prime factorization method) is provided as below:

Step 1: Find the prime factorization of each of the natural numbers

Step 2: Form the LCM of the given numbers as the product of every factor that appears in any of the prime factorization but take the highest number of times it appears.

Example 2.14: Consider the two numbers 18 and 24.

$$\begin{aligned} Step1:\ 18 &= 2^2 \times 3^2 \\ 24 &= 2^3 \times 3 \end{aligned}$$

Step 2: The factors that appear in any case are 2 and 3, but take the numbers with the highest number of times.

$\therefore \quad LCM\,(18,\ 24) = 2^3 \times 3^2 = \underline{\underline{72}}$

Example 2.15: Consider the three numbers 20, 80 and 450.

$$\begin{aligned} Step1:\ 20 &= 2^2 \times 5 \\ 80 &= 2^4 \times 5 \\ 450 &= 2 \times 3^2 \times 5^2 \end{aligned}$$

Step 2: The factors that appear in any cases are 2 , 3 and 5, but take the numbers with the highest number of times.

$\therefore \quad LCM\ (20, 80, 450) = 2^4 \times 3^2 \times 5^2 = \underline{\underline{3600}}$

2.1.1.6 Well ordering Principle in the set of natural numbers

Definition 2.7

Every non-empty subset of the set of natural numbers has smallest (least) element.

Example 2.16 $A = \{2, 3, 4, \cdots\} \subseteq N.$ $smallest\ element\ of\ A = 2.$

Note: The set of counting numbers including zero is called the set of whole numbers and is denoted by **W**. i.e **W** $= \{0, 1, 2, 3, \ldots\}$

2.1.1.7 Principle of Mathematical Induction

Mathematical induction is one of the most important techniques used to prove in mathematics. It is used to check conjectures about the outcome of processes that occur repeatedly according to definite patterns. We will introduce the technique with examples.

For a given assertion involving a natural number n, if

i. the assertion is true for n = 1 (usually).

ii. it is true for n = k+1, whenever it is true for n = k ($k \geq 1$), then the assertion is true for every natural number n.

The method is used to prove different propositions involving positive integers using three steps:

Step1: Prove that T_k (usually T_1) holds true.

Step 2: Assume that T_k for k = n is true.

Step 3: Show that T_k is true for k = n+1.

Example 2.17 Show that $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Proof:

$Step 1.$ *For* $n=1,\ 1=1^2$ *which is true.*

$Step 2.$ *Assume that it is true for* $n=k$

i.e. $1+3+5+\cdots+(2k-1)=k^2.$

$Step 3.$ *We should show that it is true for* $n=k+1.$

Claim $: 1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2$

Now $\underline{1+3+5+\cdots+(2k-1)}+(2k+1)=k^2+(2k+1)$

$$=k^2+2k+1$$

$$=\underline{\underline{(k+1)^2}} \text{ which is the required result.}$$

$\therefore$ *It is true for any natural number n.*

Example 2.18 Show that $1+2+3+\cdots+(n)=\dfrac{n(n+1)}{2}.$

Proof:

$Step 1.$ *For* $n=1,\ 1=\dfrac{1(1+1)}{2}$ *which is true.*

$Step 2.$ *Assume that it is true for* $n=k$

i.e. $1+2+3+\cdots+(k)=\dfrac{k(k+1)}{2}.$

$Step 3.$ *We should show that it is true for* $n=k+1$

Claim $: 1+2+3+\cdots+(k)+(k+1)=\dfrac{(k+1)(k+2)}{2}.$

Now $\underline{1+2+3+\cdots+(k)}+(k+1)=\dfrac{k(k+1)}{2}+(k+1)$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\underline{\underline{\frac{(k+1)(k+2)}{2}}} \text{ which is the required result.}$$

$\therefore$ *It is true for any natural number n.*

Example 2.19 Show that $5^n+6^n<9^n$ *for* $n\geq 2.$

Proof:

$Step 1.\ \ For\ n = 2,\ \ 61 < 81\ which\ is\ true$

$Step 2.\ \ Assume\ that\ it\ is\ true\ for\ n = k.$

$\quad i.e.\ 5^k + 6^k < 9^k.$

$Step 3.\ \ We\ should\ show\ that\ it\ is\ true\ for\ n = k+1$

$Claim:\ 5^{k+1} + 6^{k+1} < 9^{k+1}.$

$$Now\ 5^{k+1} + 6^{k+1} = 5.5^k + 6.6^k \leq 6.5^k + 6.6^k$$
$$= 6(5^k + 6^k)$$
$$< 9(5^k + 6^k)$$
$$< 9(9^k) = 9^{k+1}$$
$$\Rightarrow 5^{k+1} + 6^{k+1} < 9^{k+1}\quad which\ is\ the\ required\ format.$$

$\therefore\ It\ is\ true\ for\ any\ natural\ number\ n \geq 2.$

2.1.2 The set of Integers

As the knowledge and interest of human beings increased, it was important and obligatory to extend the natural number system. For instance to solve the equation x+1= 0, the set of natural numbers was not sufficient. Hence the set of integers was developed to satisfy such extended demands.

Definition 2.8

The set of integers is dented by **Z** and described as **Z** = $\{\ldots,-2, -1, 0, 1, 2, \ldots\}$

2.1.2.1 Operations on the set of integers

i) Addition (+)

If two integers *a* & *b* are added using the operation "+", then the sum *a*+*b* is also an integer. If the sum of the two integers *a* & *b* is denoted by c, then we can write the operation as: *c* = *a*+*b*, where *c* is called the sum and *a* & *b* are called terms.

Example 2.20: 4+9 = 13, here 13 is the sum whereas 4 & 9 are terms.

ii) Subtraction ($-$)

For any two integers *a* & *b*, the operation of subtracting *b* from *a*, denoted by $a-b$ is defined by $a-b=a+(-b)$. This means that subtracting *b* from *a* is equivalent to adding the additive inverse of *b* to *a*.

Example 2.21: $7-5=7+(-5)=2$

iii) Multiplication ($\times$)

If two integers a & b are multiplied using the operation "$\times$", then the product $a\times b$ is also an integer. If the product of the two integers a & b is denoted by c, then we can write the operation as: $c = a\times b$, where c is called the product and a & b are called factors.

Example 2.22: $4\times 7 = 28$, here 28 is the product whereas 4 & 7 are factors.

Properties of addition and multiplication on the set of integers

i. For any two integers a & b, the sum $a+b$ is also an integer. For instance in the above example, 4 and 9 are integers, their sum 13 is also an integer. In general, we say that the set of integers is closed under addition.

ii. For any two integers a & b, $a+b = b+a$.

Example 2.23: 4+9 = 9+4 = 13. In general, we say that addition is commutative on the set of integers.

iii. For any three integers a, b & c, $(a+b)+c = a+(b+c)$.

Example 2.24: (5+9)+8 = 5+(9+8) = 22. In general, we say that addition is associative on the set of integers.

iv. For any integer a, it holds that $a+0 = 0+a = a$.

Example 2.25: 7+0 = 0+7 = 7. In general, we say that addition has an identity element on the set of integers and 0 is the identity element.

v. For any integer a, it holds that $a+(-a)=-a+a=0$.

Example 2.26: 4+-4 = -4+4 = 0. In general, we say that every integer a has an additive inverse denoted by $-a$.

vi. For any two integers a & b, the product $a\times b$ is also an integer. For instance in the above example, 4 and 7 are integers, their product 28 is also an integer. In general, we say that the set of integers is closed under multiplication.

vii. For any two integers a & b, $a\times b = b\times a$.

Example 2.27: $4\times 7 = 7\times 4 = 28$. In general, we say that multiplication is commutative on the set of integers.

viii. For any three integers a, b & c, $(a\times b)\times c = a\times (b\times c)$.

Example 2.28: $(3\times 5)\times 4 = 3\times (5\times 4) = 60$. In general, we say that multiplication is associative on the set of integers.

ix. For any integer a, it holds that $a \times 1 = 1 \times a = a$.

Example 2.29: $5 \times 1 = 1 \times 5 = 5$. In general, we say that multiplication has an identity element on the set of integers and 1 is the identity element.

x. For any three integers a, b & c, $a \times (b+c) = (a \times b)+(a \times c)$.

Example 2.30: $4 \times (5+6) = (4 \times 5)+ (4 \times 6) = 44$. In general, we say that multiplication is distributive over addition on the set of integers.

2.1.2.2 Order Relation in Z

i) **Transitive property:** For any three integers a, b & c, $a > b$ & $b > c \Rightarrow a > c$

ii) **Addition property**: For any three integers a, b & c, $a > b \Rightarrow a + c > b + c$

iii) **Multiplication property**: For any three integers a, b and c, where c>0, $a > b \Rightarrow ac > bc$

iv) **Law of trichotomy**: For any two integers a & b we have $a > b$ *or* $a < b$ *or* $a = b$.

Exercise 2.1

1. Find an odd natural number x such that LCM (x, 40) = 1400.
2. There are between 50 and 60 number of eggs in a basket. When Loza counts by 3's, there are 2 eggs left over. When she counts by 5's, there are 4 left over. How many eggs are there in the basket?
3. The GCF of two numbers is 3 and their LCM is 180. If one of the numbers is 45, then find the second number.
4. Using Mathematical Induction, prove the following:

a) $6^n - 1$ *is divisible by* 5, *for* $n \geq 0$.

b) $2^n \leq (n+1)!$, *for* $n \geq 0$

c) $x^n + y^n$ *is divisible by* $x + y$ *for odd natural number* $n \geq 1$.

d) $2 + 4 + 6 + \cdots + 2n = n(n+1)$

e) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

f) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

g) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

2.1.3 The set of rational numbers

As the knowledge and interest of human beings increased with time, it was again necessary to extend the set of integers. For instance to solve the equation $2x+1=0$, the set of integers was not sufficient. Hence the set of rational numbers was developed to satisfy such extended needs.

Definition 2.9

Any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, is called a rational number. The set of rational numbers denoted by Q is described by

Q = $\left\{ \frac{a}{b} : a \ \ and \ \ b \ \ are \text{ integers } \ \ and \ \ b \neq 0 \right\}$.

Notes:

i. From the expression $\frac{a}{b}$, a is called numerator and b is called denominator.

ii. A rational number $\frac{a}{b}$ is said to be in lowest form if GCF $(a, b) = 1$.

2.1.3.1 Operations on the set of rational numbers

i) Addition (+)

If two rational numbers a/b and c/d are added using the operation "+", then the sum defined as $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ is also a rational number.

Example 2.31: $\frac{1}{2}+\frac{3}{5}=\frac{11}{10}$

ii) Subtraction $(-)$

For any two rational numbers a/b & c/d, the operation of subtracting c/d from a/b, denoted by a/b- c/d is defined by $a/b - c/d = a/b + (-c/d)$.

Example 2.32: $\frac{1}{2}-\frac{3}{5}=\frac{-1}{10}$

iii) Multiplication $(\times)$

If two rational numbers a/b and c/d are multiplied using the operation "$\times$", then the product defined as $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ is also a rational number.

Example 2.33: $\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}$

iv) Division ($\div$)

For any two rational numbers $a/b \,\&\, c/d$, dividing a/b by c/d is defined by

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c},\ c\neq 0.$$

Example 2.34: $\frac{1}{2}\div\frac{3}{5}=\frac{1}{2}\times\frac{5}{3}=\frac{5}{6}$

Properties of addition and multiplication on the set of rational numbers

Let a/b, c/d and e/f be three rational numbers, then

i. The set of rational numbers is closed under addition and multiplication.

ii. Addition and multiplication are both commutative on the set of rational numbers.

iii. Addition and multiplication are both associative on the set of rational numbers.

iv. **0** is the additive identity

i.e., $a/b+\mathbf{0}=\mathbf{0}+a/b=a/b$.

v. Every rational number has an additive inverse.

i.e., $a/b+(-a/b)=-a/b+a/b=\mathbf{0}$.

vi. **1** is the multiplicative identity

i.e., $a/b\times 1=\mathbf{1}\times a/b=a/b$.

vii. Every non-zero rational number has a multiplicative inverse.

i.e., $a/b\times b/a=b/a\times a/b=1$.

2.1.3.2 Order Relation in Q

i) **Transitive property**

For any three rational numbers a/b, c/d & e/f $a/b>c/d$ & $c/d>e/f \Rightarrow a/b>e/f$.

ii) **Addition property**

For any three rational numbers a/b, c/d & e/f $a/b>c/d \Rightarrow a/b+e/f>c/d+e/f$.

iii) **Multiplication property**

For any three rational numbers a/b, c/d, e/f and $e/f > 0$

$a/b > c/d \;\; \Rightarrow (a/b)(e/f) > (c/d)(e/f)$.

iv) **Law of trichotomy**

For any two rational numbers a/b & c/d we have $a/b > c/d$ *or* $a/b < c/d$ *or* $a/b = c/d$.

2.1.3.3 Decimal representation of rational numbers

A rational number $\frac{a}{b}$ can be written in decimal form using long division.

Terminating decimals

Example 2.35: Express the fraction number $\frac{25}{4}$ in decimal form.

$Solution: \frac{25}{4} = 6.25$

Non-terminating periodic decimals

Example 2.36: Express the fraction number $\frac{25}{3}$ in decimal form.

$Solution: \frac{25}{3} = 8.333\cdots$

Now we will see how to convert decimal numbers in to their fraction forms. In earlier mathematics topics, we have seen that multiplying a decimal by 10 pushes the decimal point to the right by one position and in general, multiplying a decimal by 10^n pushes the decimal point to the right by n positions. We will use this fact for the succeeding topics.

2.1.3.4 Fraction form of decimal numbers

A rational number which is written in decimal form can be converted to a fraction form as $\frac{a}{b}$ in lowest (simplified) form, where a and b are relatively prime.

Terminating decimals

Consider any terminating decimal number d. Suppose d terminates n digits after the decimal point. d can be converted to its fraction form as below:

$$d = d \times 1 = d \times \frac{1}{1} = \underline{\underline{d \times (\frac{10^n}{10^n})}}.$$

Example 2.37: Convert the terminating decimal 3.47 to fraction form.

$Solution:\ 3.47 = 3.47 \times \frac{10^2}{10^2} = \frac{347}{100}.$

Non-terminating periodic decimals

Consider any non-terminating periodic decimal number d. Suppose d has **k** non-terminating digits and **p** terminating digits after the decimal point. d can be converted to its fraction form as below:

$$d = d \times 1 = d \times \frac{1}{1} = d \times (\frac{10^{k+p} - 10^k}{10^{k+p} 10^k}).$$

Example 2.38: Convert the non-terminating periodic decimal $42.5\overline{38}$ to fraction form.
Solution: k = 1, p = 2.

$$\therefore\ d = d \times 1 = d \times \frac{1}{1} = d \times (\frac{10^{k+p} - 10^k}{10^{k+p} - 10^k}) = 42.5\overline{38} \times (\frac{10^3 - 10}{10^3 - 10}) = \frac{42538.\overline{38} - 425.\overline{38}}{1000 - 10} = \frac{42113}{990}.$$

Note: From the above two cases, we can conclude that both terminating decimals and non-terminating periodic decimals are rational numbers. (Why? Justify).

2.1.3.5 Non-terminating and non-periodic decimals

Some decimal numbers are neither terminating nor non-terminating periodic. Such types of numbers are called irrational numbers.

Example 2.39: 62.757757775….

Example 2.40: Show that $\sqrt{2}$ is an irrational number.

Proof:

$Suppose\ \sqrt{2}\ is\ a\ rational\ number$

$\Rightarrow \sqrt{2} = \frac{a}{b},\ where\ GCF(a,b) = 1$

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2 ....(*)$

$\Rightarrow a^2\ is\ even$

$\Rightarrow a\ is\ even$

$\Rightarrow a = 2n\ ....(**)$

Putting this in (*) *we get* :

$\Rightarrow 4n^2 = 2b^2$

$\Rightarrow b^2 = 2n^2$

$\Rightarrow b^2$ *is even*

$\Rightarrow b$ *is even*

$\Rightarrow b = 2m$............(***)

From (**) and (***) we get a contradiction that GCF (a, b) = 1 which implies that $\sqrt{2}$ is not a rational number.

Therefore, $\sqrt{2}$ is an irrational number.

2.1.4 The set of real numbers

Definition 2.10

A number is called a real number if and only if it is either a rational number or an irrational number.

The set of real numbers denoted by $\Re$ can be described as the union of the set of rational and irrational numbers. i.e $\Re$ = {x : x is a rational number or an irrational number}.

There is a 1-1 correspondence between the set of real numbers and the number line (For each point in the number line, there is a corresponding real number and vice-versa).

2.1.4.1 Operations on the set of real numbers

i) Addition (+)

If two real numbers are added using the operation "+", then the sum is also a real number.

ii) Subtraction ($-$)

For any two real numbers a & b, the operation of subtracting b from a, denoted by $a - b$ is defined by $a - b = a + (-b)$.

iii) Multiplication ($\times$)

If two real numbers a and b are multiplied using the operation "$\times$", then the product defined as $a \times b = ab$ is also a real number.

iv) Division ($\div$)

For any two real numbers a & b, dividing a by b is defined by $a \div b = a \times \frac{1}{b}$, $b \neq 0$.

Properties of addition and multiplication on the set of real numbers

Let a, b and c be three real numbers, then

i. The set of real numbers is closed under addition and multiplication.

ii. Addition and multiplication are commutative on the set of real numbers.

iii. Addition and multiplication are associative on the set of real numbers.

iv. **0** is the additive identity

i.e., $a + \mathbf{0} = \mathbf{0} + a = a$.

v. Every real number has an additive inverse.

i.e., $a + (-a) = -a + a = \mathbf{0}$.

vi. **1** is the multiplicative identity

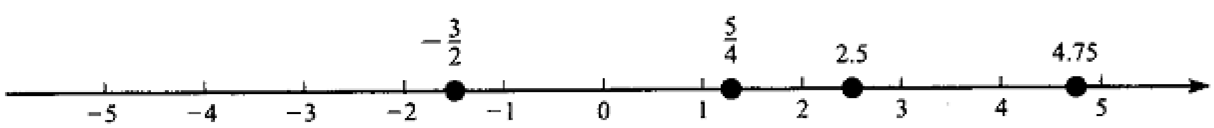
i.e., $a \times 1 = \mathbf{1} \times a = a$.

vii. Every non-zero real number has a multiplicative inverse.

i.e., $a \times 1/a = 1/a \times a = 1$.

2.1.4.2 The real number and the number line

One of the most important properties of the real number is that it can be represented graphically by points on a straight line. The point 0 is termed as the origin. Points right of 0 are called positive real numbers and points let of 0 are called negative real numbers. Each point on the number line corresponds a unique real number and vice-versa.



Geometrically we say a is greater than b if a is located to the right of b on the nuber line.

2.1.4.3 Order Relation in R

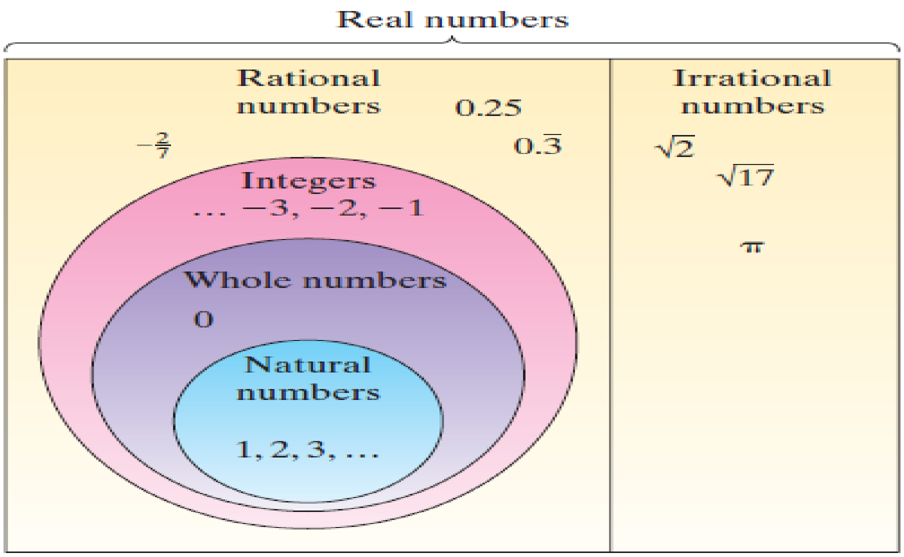
i) **Transitive property:** For any three real numbers a, b & c, $a > b$ & $b > c \Rightarrow a > c.$

ii) **Addition property:** For any three real numbers a, b & c, $a > b \Rightarrow a + c > b + c.$

iii) **Multiplication property:** For any three real numbers a, b, c and $c > 0$, we have $a > b \Rightarrow ac > bc.$

iv) **Law of trichotomy:** For any two real numbers a & b we have $a > b$ *or* $a < b$ *or* $a = b.$

Summary of the real number system



2.1.4.4 Intervals

Let a and b be two real numbers such that $a<b$, then the intervals which are subsets of **R** with end points a and b are denoted and defined as below:

i. $(a,b)=\{x: a<x<b\}$ open interval from a to b.

ii. $[a,b]=\{x: a\le x\le b\}$ closed interval from a to b.

iii. $(a,b]=\{x: a<x\le b\}$ open-closed interval from a to b.

iv. $[a,b)=\{x: a\le x<b\}$ closed-open interval from a to b.

2.1.4.5 Upper bounds and lower bounds

Definition 2.11

Let A be non – empty and $A\subseteq \Re$.

i. A point $a\in R$ is said to be an upper bound of A iff $x\le a$ for all $x\in A$.

ii. An upper bound of A is said to be least upper bound (lub) iff it is the least of all upper bounds.

iii. A point $a\in R$ is said to be lower bound of A iff $x\ge a$ for all $x\in A$.

ii. A lower bound of A is said to be greatest lower bound (glb) iff it is the greatest of all lower bounds.

Example 2.41 *Consider the set* $A=[2,\ 5)\subseteq \Re$.

$i)$ *lower bounds are* $\cdots, -9, -3, 0, \frac{1}{2}, 1, 2$

Here the greatest element is $2.$

$\therefore \ \text{glb} = 2$

$ii)$ *upper bounds are* $5, 6, \frac{25}{3}, 20, 99, 1000\cdots$

Here the least element is $5.$

$\therefore \ \text{lub} = 5.$

Example 2.42: Consider the set **A =** $\left\{\frac{1}{n}\right\}$ for n$\in N$.

Solution: $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$

$i)$ *lower bounds are* $\cdots, -3, -2, 0$

Here the greatest element is $0.$ *Thus*, $\text{glb} = 0$

$ii)$ *upper bounds are* $1, 3, \frac{9}{2}, 50, \cdots$

Here the least element is $1.$ *Thus*, $\text{lub} = 1.$

Based on the above definitions, we can define the completeness property of real numbers as below.

2.1.4.6 Completeness property of real number (R)

Completeness property of real numbers states that: Every non-empty subset of $\Re$ that has lower bounds has glb and every non-empty subset of $\Re$ that has upper bounds has a lub.

Exercise 2.1

1. Express each of the following rational numbers as decimal:

   $a)\ \frac{4}{9}$ $\quad b)\ \frac{3}{25}$ $\quad c)\ \frac{11}{7}$ $\quad d)\ -5\frac{2}{3}$ $\quad e)\ \frac{2}{77}$

2. Write each of the following as decimal and then as a fraction:

   a) three tenths b) four thousands

3. Write each of the following in meters as a fraction and then as a decimal

   a) 4mm b) 6cm and 4mm c) 56cm and 4mm

4. Classify each of the following as terminating or non-terminating periodic

$a)\ \frac{5}{13}$ $b)\ \frac{7}{10}$ $c)\ \frac{69}{64}$ $d)\ \frac{11}{60}$ $e)\ \frac{5}{12}$

5. Convert the following decimals to fractions:

   $a)\ 3.2\bar{5}$ $b)\ 0.3\overline{14}$ $c)\ 0.\overline{275}$

6. Determine whether the following are rational or irrational:

   $a)\ 2.7\bar{5}$ $b)\ 0.272727\cdots$ $c)\ \sqrt{8}-\frac{1}{\sqrt{2}}$

7. Which of the following statements are true and which of them are false?
   a. The sum of any two rational numbers is rational
   b. The sum of any two irrational numbers is irrational
   c. The product of any two rational numbers is rational
   d. The product of any two irrational numbers is irrational
8. Find two rational numbers between

2.2. Complex Number

Introduction

Solving algebraic equations has been historically one of the favorite topics of mathematicians. While linear equations are always solvable in real number, but not quadratic equations have this property; for instance $x^2 + 1 = 0$ has no real solution. Until the 18th century mathematicians avoid quadratic equation that were not solvable over real number. LEONHARD EULER broke this idea and introducing the number $\sqrt{-1}$ and denotes this number by i and called an imaginary unit. This becomes one of the most useful symbols in mathematics and useful symbol to define complex number. The study of complex numbers continues and has been enhanced in the last two and half centuries, in fact it is impossible to imagine modern mathematics without complex number.

Our main goal is to introduce about complex number, the unit runs smoothly between key concepts and elementary results concerning complex numbers, the student has the opportunity to learn how complex numbers can be employed in solving algebraic equations and to understand the geometric interpretation of complex number and the operations involving them.

Definition:-A complex number z is given by a pair of real numbers x and y and written as

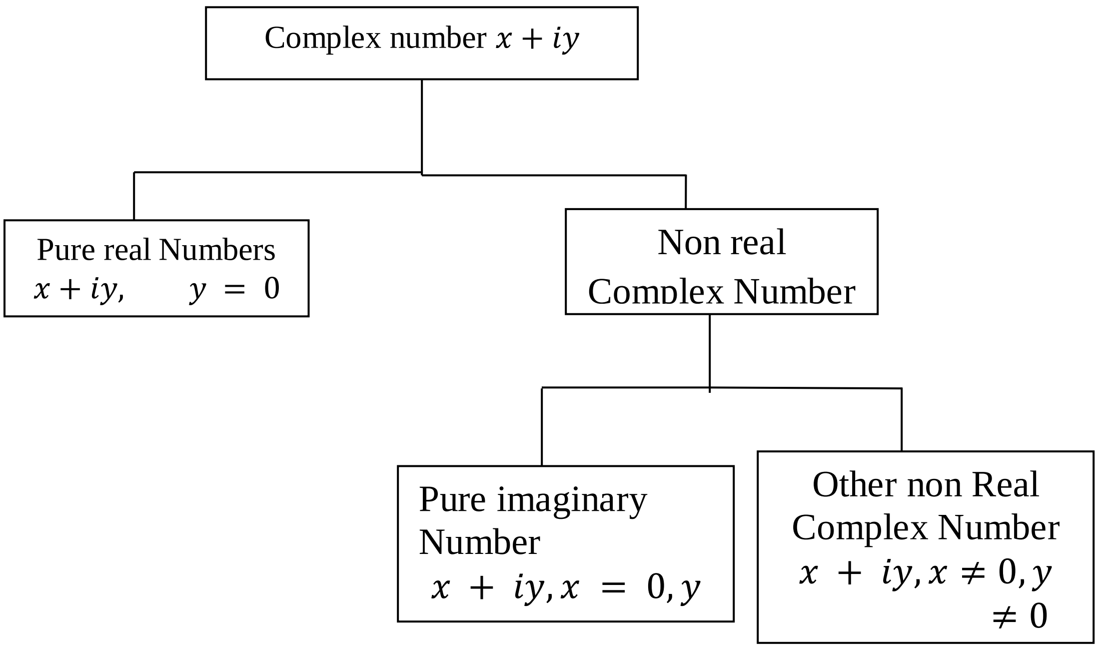
$z = x + iy$ which is Cartesian form of a complex number and denoted by $\mathbb{C}$.

$$\mathbb{C} = \{x + yi: x \text{ and } y \text{ are real numbers and } i = \sqrt{-1}\}$$

Where, $i = \sqrt{-1} \Longrightarrow i^2 = -1$,$x$ is a real part of z and denoted by $\boldsymbol{Rez}$ and y is the imaginary part of z and denoted by $\boldsymbol{Imz}$.

If y = 0 then the complex number is purely real number and if x = 0then the number is purely imaginary.

The figure below illustrates the relationships between complex numbers



The Argand Diagram

The complex number may be represented as point in the coordinate plane sometimes called the **Argand Diagram**. The real number 1 is represented by the point (1, 0) and the complex number i is represented by the point (0, 1). In general the complex number $z = x + iy$ represented by the point (x, y) on the coordinate plane, where the horizontal axis (x-axis) represent a complex number $x + iy$ with $y = 0$, we call this horizontal axis **the real axis** and the vertical axis (y-axis) represent a complex number $x + iy$ with $x = 0$, we call this axis **the imaginary axis.** These two perpendicular axes intersect at a point ‘O’ which is called the **origin**. The set of real number was extended to the set of complex numbers so that real number is the proper subset of complex number.

2.2.1 Operations on Complex Number

i. Addition, Subtraction and Multiplication of Complex Number

There are two methods to determine the sum and difference of two complex number; these are

- Algebraic method.
- Graphical method using the Argend Diagram.

Algebraic Method

$If z_1 = x_1 + iy_1 and z_2 = x_2 + iy_2$ Then,

- $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$
- The sum and difference of complex numbers algebraically

$$Z_1 = z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$$

$$\boxed{\boldsymbol{Z_1 = (x_1 + x_2) + i(y_1}}$$

$$Z_2 = z_1 - z_2 = x_1 + iy_1 - (x_2 + iy_2)$$

$$\boxed{\boldsymbol{Z_2 = (x_1 - x_2) + i(y_1}}$$

The real terms are added or subtracted and the imaginary terms are added and subtracted separately.

- The product of complex numbers algebraically

$$Z_3 = kz_1 = kx_1 + iky_1, for all k \in \mathbb{R}$$

$$\boxed{\boldsymbol{Z_3 = kz_1 = kx_1 + iky_1}}$$

$$Z_4 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1x_2 + x_1iy_2 + iy_1x_2 + iy_1iy_2$$

$$= x_1x_2 + i(x_1y_2 + y_1x_2) + i^2y_1y_2$$

$$\boxed{\boldsymbol{Z_4 = z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2}}$$

$$\boxed{\begin{array}{c} \boldsymbol{ReZ_4 = x_1x_2 - y_1y_2} \\ \boldsymbol{ImZ_4 = x_1y_2 + y_1x_2} \end{array}}$$

Example; operate the following

a. $(4 + 7i) + (1 - 6i) = (4 + 1) + (7 - 6)i = 5 + i$

b. $(1 + 2i) - (4 + 2i) = (1 - 4) + (2 - 2)i = -3 + 0i = -3$

c. $(1+i)(\sqrt{3}-i) = 1*\sqrt{3} + 1*-i + i*\sqrt{3} + i*-i$

$$= \sqrt{3} - i + \sqrt{3}i - i^2 = \sqrt{3} - -1 + (\sqrt{3}-1)i$$

$$= (\sqrt{3}+1) + (\sqrt{3}-1)i$$

Activity 2.1

Express each of the following in the form of $x + iy$

a. $1 - i + 4 + 3id.$ $(2+3i) - (3-5i) + (4+3i)$

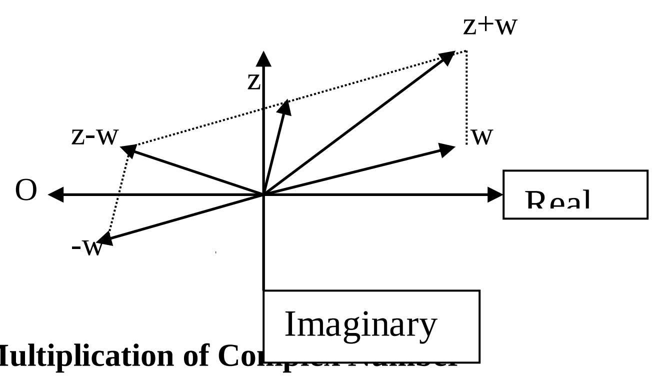
i. Graphical method using the Argend Diagram

If we have two complex numbers $z = x + iy$ and $w = u + iv$ then their sum $z + w$ and difference $z - w$ are given by

$$z + w = (x+u) + i(y+v)$$
$$z - w = (x-u) + i(y-v)$$

And therefore appears on the Argand Diagram as the vector sum of z and w

The complex number $z + w$ is represented geometrically as the fourth vertex of the parallelogram formed by O, z, and w,



Properties of Addition and Multiplication of Complex Number

Addition and multiplication of complex numbers satisfies the following properties

- **Commutative law**

$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 x z_2 = z_2 x z_1 \end{cases} forall\ z_1, z_2 \in \mathbb{C}$$

- **Associative law**

$$\begin{cases} (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \\ (z_1 x z_2) x\ z_3 = z_1 x\ (z_2 x\ z_3) \end{cases} forall\ z_1, z_2,\ z_3 \in \mathbb{C}$$

- **Additive and Multiplicative identity**

For any complex number $z = (x, y) \in \mathbb{C}$,there are unique complex numbers

$$1 = (1,0) and\ 0 = (0,0) \in \mathbb{C},$$

Such that,

$$\begin{cases} z + 0 = 0 + z = z \\ z * 1 = 1 * z = z \end{cases} \text{for all } z \in \mathbb{C}$$

- **Additive and Multiplicative inverse**

For any complex number $z = (x, y) \in \mathbb{C}$ there are unique complex numbers

$$-z = (-x, -y) and z^{-1} = (x^{-1}, y^{-1}) \in \mathbb{C},$$

Such that,

$$\begin{cases} z + (-z) = (-z) + z = 0 \\ z \text{ x } z^{-1} = z^{-1} \text{x } z = 1 \end{cases}$$

To find $z^{-1} = \frac{1}{z} = (x^{-1}, y^{-1})$,

Observe that $z = x + iy \neq 0 \Longrightarrow x \neq 0 \; or y \neq 0, equivalently x^2 + y^2 \neq 0$

The relation $z \text{ x } z^{-1} = 1, gives$

$$(x + iy) * \left(x^{-1} + iy^{-1}\right) = 1 + 0i$$

$$\left(xx^{-1} - yy^{-1}\right) + i\left(yx^{-1} + xy^{-1}\right) = 1 + 0i, equivalently$$

$$\begin{cases} xx^{-1} - yy^{-1} = 1 \\ yx^{-1} + xy^{-1} = 0 \end{cases}$$

Solving the system with respect to $x^{-1} and y^{-1}$ we obtain

$$x^{-1} = \frac{x}{x^2 + y^2} and y^{-1} = \frac{-y}{x^2 + y^2}$$

Hence, the multiplicative inverse of the complex number $z = x + iy$ is

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Example; Let z = 1+2i then Find the additive and multiplicative inverse of z

Solution; x = 1, y = 2 and $x^2 + y^2 = 1^2 + 2^2 = 1 + 4 = 5$, then

$$x^{-1} = \frac{x}{x^2 + y^2} = \frac{1}{5} and y^{-1} = \frac{-y}{x^2 + y^2} = \frac{-2}{5}$$

$z^{-1} = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{-y}{x^2+y^2} = \frac{1}{5} - i\frac{2}{5}$ and $-1 - 2i$ are multiplicative and additive inverse of $z = 1 + 2i$, respectively.

- **Distributive law**

$$(z_1 \text{x}(z_2 + z_3) = z_1 \text{x} z_2 + z_1 \text{x} z_3. \;\; for all \; z_1, z_2, \; z_3 \in \mathbb{C}$$

Activity 2.2

Find the additive and multiplicative inverse of -2 -3i

2.2.2 The Conjugate and Modulus of Complex Number

The Conjugate of Complex Number

Let $z = x + iy$ where $x, y \in \mathbb{R}$ the conjugate of z is denoted by $\bar{z}$ and is equal to $\bar{z} = x - iy$.

$\Longrightarrow$The conjugate of $z = -x - iy$ is $\bar{z} = -x + iy$.

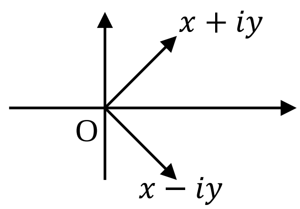
Note:

- The product of a complex number and its conjugate is always real and positive.

$$(x - iy)(x + iy) = x^2 + y^2$$

- If z is real, $\bar{z} = z$, i.e; z is its own conjugates

It is necessary to represent these complex numbers in an Argand Diagram. The conjugate of $z = x + iy$ is the reflection of z on the x – axis.



Example: multiply each complex number by its conjugate

a. $1 + ib$. $4 - 3i$

Solution;

a.$(1 + i)(1 - i) = 1x1 + 1x(-i) + ix1 + i(-i) \ = 1 - i + i - i^2 = 1 - (-1) \ = 2$

b.$(4 - 3i)(4 + 3i) = 4x4 + 4x3i + (-3i)x4 + (-3i)(3i) = 16 - i^2 9 = 16 + 9 = 25$

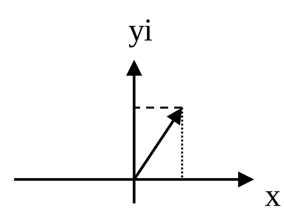
Activity 2.3

Multiply complex number3i (3+5i) by its conjugate

2.2.3 The Modulus of Complex Number

The modulus (absolute value) $|z|$of a complex number $z = x + iy$ is its distance from the origin. If $z = x + iy$,then

$$|z| = \sqrt{x^2 + y^2} \Longrightarrow |z|^2 = x^2 + y^2 = z\bar{z}$$



2.2.4 Quotient of Complex Number

When the complex number is expressed as a quotient which contains i in the denominator; it is necessary to multiply a quotient complex expression by the quotient conjugate of the denominator in order to obtain a real quantity in the denominator.

Let z be the quotient of two complex numbers

$$Z = \frac{x_1 + iy_1}{x_2 + iy_2} \text{--}(*)$$

It required to express the complex number in the form of$a + bi$, where a and b are real, multiply (*) by the quotient conjugate of$x_2 + iy_2$, we have;

$$Z = \frac{x_1 + iy_1}{x_2 + iy_2} . \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1x_2 + ix_2y_1 - iy_2x_1 - i^2y_1y_2}{{x_2}^2 - (iy)^2}$$

$$= \frac{x_1x_2 + i(x_2y_1 - y_2x_1) + y_1y_2}{{x_2}^2 + {y_2}^2}$$

$$\boldsymbol{Z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{{x_2}^2 + {y_2}^2} + \frac{x_2y_1 - y_2x_1}{{x_2}^2 + {y_2}^2} i}$$

$$\Longrightarrow \mathbf{ReZ} = a = \frac{x_1x_1 + y_1y_2}{{x_2}^2 + {y_2}^2} and \mathbf{ImZ} = b = \frac{x_2y_1 - y_2x_1}{{x_2}^2 + {y_2}^2}$$

Square Root of a Negative Number

For any positive real number "a"

$$\sqrt{-a} = \sqrt{-1}\sqrt{a} = i\sqrt{a} \text{ is called the principal squareroot of } -a$$

With this convention the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even $b^2 - 4ac < 0$

In general if we allow complex number as solution, any quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b and c has solutions and these solutions are always complex conjugates of each other. It is also true that every polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + - - -a_1x + a_0 = 0$$

of degree at least two has a solution among the complex number.

Example: Find the roots of the equation

$$x^2 + x + 1 = 0$$

Solution: $a = 1, b = 1 \; and c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$x = \frac{-1}{2} + i\frac{\sqrt{3}}{2} or x = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

are solutions of $x^2 + x \; + 1 = 0$

Activity 2.4

Solve $3x^2 - 2x + 5 = 0$

Note: when simplifying or performing operations involving radicals with a negative radicand and even index, it is important to write the numbers in terms of the imaginary unit i, if possible. The property, $\sqrt{a}$ x $\sqrt{b} = \sqrt{a\text{x}b}$ is only true when $a \geq 0 \; and b \geq 0$. This property does not apply to non-real numbers. To find the correct answer, if a and b are negative we must first write each number in terms of the imaginary unit i.

Example: $\sqrt{-3}\text{x}\sqrt{-4} = \sqrt{(-3)\text{ x }(-4)} = \sqrt{12} = 2\sqrt{3}$

is not true, but it is true that

$$\sqrt{-3} * \sqrt{-4} = i\sqrt{3} * i\sqrt{4} = i^2 2\sqrt{3} = -2\sqrt{3}$$

Fundamental Properties of $|z| and \bar{z}$

i. $|z|^2 = x^2 + y^2 = z\bar{z}$

$If z \; = \; x + iy then$

$$|z| = \sqrt{x^2 + y^2} \Longrightarrow |z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}$$

$$\boldsymbol{|z|^2 = z\bar{z}}$$

ii. $Re z = \dfrac{z + \bar{z}}{2} and Im z = \dfrac{z - \bar{z}}{2i}$

$If z = x + iy then$

$$z + \bar{z} \; = \; (x + iy) + (x - iy) = 2x z - \bar{z} = (x + iy) - (x - iy)$$

$$x = \frac{z + \bar{z}}{2} 2iy = \; z - \bar{z} \Longrightarrow y = \frac{z - \bar{z}}{2i}$$

$$\Longrightarrow \boldsymbol{Re = \frac{z + \bar{z}}{2}} \Longrightarrow \boldsymbol{Im z = \frac{z - \bar{z}}{2i}}$$

iii. $\overline{z + w} = \bar{z} + \bar{w}$

$If z \; = \; x + iy and w \; = \; u + iv then$

$z + w \; = \; (x + iy) \; + \; u + iv \; = \; (x + u) + i(y + v)$

$\overline{z+w} = (x+u) - i(y+v) = (x+u) - i(y+v) = x + u - iy - iv$

$= x - iy + u - iv = \bar{z} + \bar{w}.$

$$\boldsymbol{\overline{z+w} = \bar{z} + \bar{w}}$$

$\boldsymbol{iv}.\ \overline{zw} = \bar{z}\bar{w} If z = x + iy and w = u + iv\ then$

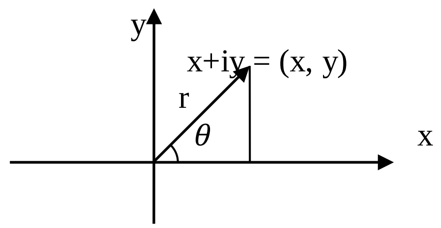
Zw = $(x+iy)(u+iv) = (xu - yv) + i(xv + yu)$ $\bar{z}\bar{w} = (x - iy)(u - iy)$

$\overline{zw} = (xu - yv) - i(xv + yu)$ $= (xu - yv) - i(xv + yu)$

$$\Rightarrow \boldsymbol{\overline{zw} = \bar{z}\bar{w}}$$

2.4 Polar Form of Complex Numbers

We know that any complex number $z = x + iy$ can be considered as a point (x, y) and any such point can be represented by polar coordinates (r, θ) with $r \geq 0$



$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$z = x + iy = r\sin\theta + ir\cos\theta = r(\cos\theta + isin\theta)$$

$$\Rightarrow z = r(\cos\theta + isin\theta)\ is\ polar\ form\ of a\ complex\ number\ z = x + iy$$

Where r $= |z| = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$ and θ is an angle formed by a vector of complex number $z = x + iy$ Through the origin and measured with the reference to the positive x-axis in an anticlockwise direction.

The angle θ is called the **argument of z** and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique, any two argument of z differ by an integer multiple of 2π $(i.e \arg(z) = \theta + 2n\pi, n \in \mathbb{R})$. But the principal argument of z is unique which is an argument of z that lies between $-\pi\ and\ \pi$ and denoted by Arg(z).

$$i.e - \pi \leq Arg(z) \leq \pi$$

Example: Write the following numbers in polar form

a. $1 + i$ b. $\sqrt{3} - i$

Solution; a. $1+i$

r= $\sqrt{x^2+y^2} = \sqrt{(1)^2+(1)^2} = \sqrt{2}$ and

$$\tan\theta = \frac{y}{x} \Longrightarrow \theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1}{1}\right) = \arctan(1) = 45^0 = \frac{\pi}{4}$$

$$z = r(\cos\theta + isin\theta) = \sqrt{2}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) ispolarformofz = 1+i$$

b. $\sqrt{3} - i$ (**exercise**)

The polar form of complex number gives insight into multiplication and division.

Let $z_1 = r_1(\cos\theta_1) + isin\theta_1)$ and $z_2 = r_2(\cos\theta_2) + isin\theta_2)$ are two complex numbers written in polar form, then

$$z_1z_2 = (r_1(\cos\theta_1 + isin\theta_1))(r_2(\cos\theta_2 + isin\theta_2))$$

$$= r_1r_2(\cos\theta_1 + isin\theta_1)(\cos\theta_2 + isin\theta_2)$$

$$= r_1r_2[(\cos\theta_1\cos\theta_2 - sin\theta_1 sin\theta_2) + i(sin\theta_1 cos\theta_2 + sin\theta_2 cos\theta_1)]$$

$$= r_1r_2[\cos(\theta_1+\theta_2) + isin\,(\theta_1+\theta_2)]$$

$$\boldsymbol{z_1\, z_2 = r_1r_2[\cos(\theta_1+\theta_2) + isin\,(\theta_1+\theta_2)]}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + isin\theta_1)}{r_2(\cos\theta_2 + isin\theta_2)} = \frac{r_1}{r_2}\frac{\cos\theta_1 + isin\theta_1}{\cos\theta_2 + isin\theta_2} x \frac{\cos\theta_2 - isin\theta_2}{\cos\theta_2 - isin\theta_2}$$

$$= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1 + isin\theta_1)x(\cos\theta_2 - isin\theta_2)}{(\cos\theta_2 + isin\theta_2)x(\cos\theta_2 - isin\theta_2)}\right]$$

$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1 cos\theta_2 - \cos\theta_1 isin\theta_2 + isin\theta_1\cos\theta_2 - isin\theta_1 isin\theta_2}{(\cos\theta_2)^2 - (i\sin\theta_2)^2}\right]$$

$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1 cos\theta_2 - i^2 sin\theta_1 sin\theta_2 + i(sin\theta_1 cos\theta_2 - cos\theta_1 sin\theta_2}{cos^2\theta_2 + sin^2\theta_2}\right]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + isin\,(\theta_1-\theta_2)]$$

$$\boldsymbol{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + isin\,(\theta_1-\theta_2)]}$$

In particular let$z_1 = 1\ andz_2 = z, then\theta_1 = 0,\ \ \theta_2 = \theta$, we have the following

If $z = r(cos\theta + isin\theta)$ then

$$\boldsymbol{\frac{1}{z} = \frac{1}{r}(\cos\theta - isin\theta)}$$

Activity 3.5

1. Write the complex number -1 in polar form.
2. Write the complex number $8(\cos135^0 + i\sin135^0)$ in rectangular form

Exponential form of Complex Number

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Therefore,

$$z = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

Where r is the modulus of z and θ is the argument of z.

Powers of Complex Numbers

The trigonometric form of a complex number is used to raise a complex number to a power. To accomplish this, consider repeated use of the multiplication rule.

$$z = r(\cos\theta + i\sin\theta)$$

$$z^2 = r(\cos\theta + i\sin\theta) \text{ x } r(\cos\theta + i\sin\theta)$$

$$= r^2(\cos2\theta + i\sin2\theta)$$

$$z^3 = r^2(\cos2\theta + i\sin2\theta) \text{ x } r(\cos\theta + i\sin\theta)$$

$$= r^3(\cos3\theta + i\sin3\theta)$$

$$z^4 = r^4(\cos4\theta + i\sin4\theta)$$

$$z^5 = r^5(\cos5\theta + i\sin5\theta)$$

This pattern leads to **DeMoivre's Theorem**

If $z = r(\cos\theta + i\sin\theta)$ and n is any positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Activity 2.6

Compute $(\frac{1+i}{\sqrt{2}})^8$

1.6.2, Square Root of Complex Number

Let z = x+ iy, then the square root of z = x+iy is denoted by $\sqrt{x+iy}$ and

Let, $\sqrt{x+iy} = \pm(a+ib)$, then squaring on both sides, we get

$x + iy = (a + ib)^2 = a^2 - b^2 + 2abi$

$$\Rightarrow x = a^2 - b^2 and y = 2ab - - - - - - - - - - - - -(1)$$

$$b^2 = a^2 - x, ----------------------(*)$$

$$(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2 = x^2 + y^2$$

$$\Longrightarrow a^2 + b^2 = \sqrt{x^2+y^2} ----------------(2)$$

$$b^2 = \sqrt{x^2+y^2} - a^2, ------------------(**)$$

From $(*)$and $(**)$

$$a^2 - x = \sqrt{x^2+y^2} - a^2$$

$$2a^2 = \sqrt{x^2+y^2} + x$$

$$a^2 = \frac{\sqrt{x^2+y^2}-x}{2},$$

$$\boldsymbol{a = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}}}$$

From (1) and (2)

$$-\begin{cases} a^2 + b^2 = \sqrt{x^2+y^2} \\ a^2 - b^2 = \qquad x \end{cases}$$

$$2b^2 = \sqrt{x^2+y^2} - x$$

$$b^2 = \frac{\sqrt{x^2+y^2}-x}{2},$$

$$\boldsymbol{b = \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}}$$

Therefore

$$\sqrt{x+iy} = \pm(a+ib) = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$$

Example; find the square root of $3+4i$
Solution; $x = 3, y = 4$

$$let\sqrt{3+4i} = \pm(a+ib)$$

$$a = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} = \sqrt{\frac{\sqrt{3^2+4^2}+3}{2}} = \sqrt{\frac{\sqrt{9+16}+3}{2}} = \sqrt{\frac{5+3}{2}} = \sqrt{4} = 2$$

$$b = \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} = \sqrt{\frac{\sqrt{3^2+4^2}-3}{2}} = \sqrt{\frac{\sqrt{9+16}-3}{2}} = \sqrt{\frac{5-3}{2}} = \sqrt{1} = 1$$

$$\Longrightarrow \sqrt{3+4i} = \pm(a+ib) = \pm(2+i)$$

Activity 2.7

Verify that $3-7i$ is one of the square roots of $-40-42i$

The n^th Roots of a complex Number

Defining the n^{th} roots of a complex number, consider a positive integer $n \geq 2$ and a complex number $z_0 \neq 0$. As in the field of real numbers, the equation

$$Z^n - z_0 = 0 ------------------(\Delta)$$

is used for defining the n^{th} roots of the complex number z_0. Hence we call any solution Zof the equation (Δ) is an n^{th}root of the complex number z_0.

Theorem;

Let $z_0 = r(\cos\theta + i\sin\theta)$be a complex number with r >0 and $\theta \in [0, 2\pi)$.The number $z_0$has n distinct n^{th} roots, given by the formulas

$$Z_k = \sqrt[n]{r}[\cos(\frac{\theta+2k\pi}{n}) + isin\,(\frac{\theta+2k\pi}{n})], k = 0,1,2,\ldots,n-1$$

Proof; we use the polar representation of the complex number Z with the extended argument$\phi andmodulusofZis\rho.$

$$Z = \rho(cos\phi + isin\phi).$$

By definition, we have $Z^n = z_0$ or equivalently

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

We obtain $\rho^n = r$ and $n\phi = \theta + 2k\pi$, for $k \in Z$; hence $\rho = \sqrt[n]{r}$ and $\phi = \frac{\theta+2k\pi}{n}$, for $k \in Z$.

So far the roots of equation (Δ) are

$$Z_k = \sqrt[n]{r}(\cos\phi_k + i\sin\phi_k) = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

Now observe that $0 \leq \phi_0 < \phi_1 < \cdots < \phi_{n-1} < 2\pi$, so the numbers ϕ_k, where $k \in \{0, 1, \ldots, n-1\}$, are reduced arguments. Until now we had n distinct roots of Z. Such as;$Z_0, Z_1, \ldots, Zn-1$.

Example, Let us find the third roots of the number $z = 1 + i$.

Solution; The polar representation of $z = 1 + i$ is

$$x = 1, y = 1, r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}and$$

$$\text{Arg}(z) = \theta = arctan\frac{y}{x} = \arctan\left(\frac{1}{1}\right) = \arctan 1 = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$$

$$Z_k = \sqrt[3]{\sqrt{2}}\left(\cos(\frac{\pi}{3*4} + k\frac{2\pi}{3}) + i\sin(\frac{\pi}{3*4} + k\frac{2\pi}{3})\right)$$

$$= \sqrt[6]{2}\left(\cos(\frac{\pi}{12} + k\frac{2\pi}{3}) + i\sin(\frac{\pi}{12} + k\frac{2\pi}{3})\right)$$

For, k =0, 1, 2

$$Z_0 = \sqrt[6]{2}\left(\cos(\frac{\pi}{12} + 0*\frac{2\pi}{3}) + i\sin(\frac{\pi}{12} + 0*\frac{2\pi}{3})\right) \quad = \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$Z_1 = \sqrt[6]{2}\left(\cos(\frac{\pi}{12} + \frac{1\text{x}2\pi}{3}) + i\sin\left(\frac{\pi}{12} + \frac{1\text{x}2\pi}{3}\right)\right) = \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$Z_2 = \sqrt[6]{2}\left(\cos(\frac{\pi}{12} + \frac{2\text{x}2\pi}{3}) + i\sin\left(\frac{\pi}{12} + \frac{2\text{x}2\pi}{3}\right)\right) = \sqrt[6]{2}\left(\cos(\frac{\pi}{12} + \frac{4\pi}{3}) + i\sin(\frac{\pi}{12} + \frac{4\pi}{3})\right)$$

$$= \sqrt[6]{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

Exercise 2.2

1. Express each of the following in the form of $x + iy$
   a. $(1+i)(3-4i)$ c. $6i(\frac{2}{3} + \frac{5}{6}i)$
   b. $2(1+4i) - 3(2+i)$ d. $(5-2i)^2$
2. Find the additive and multiplicative inverse of
   a. 4 + 3i b. 5 – 3i
3. multiply each complex number by its conjugate
   a. $\frac{17}{25} + \frac{6}{25}i$ b. $i\sqrt{21}$
4. solve x^2+4=0
5. Write each of the following in polar form.
   a. 3 -3i b, $\frac{1}{2} - i\frac{1}{2}\sqrt{3}$
6. Write each of the following in rectangular form
   a. 4(cos30^0+$i$sin30^0) c. $\frac{1}{2}(\cos 150^0 + isin150^0)$
   b. $\sqrt{2}(cos225^0 + isin225^0)$ d. 5(cos450^0+ $i$sin450^0)
7. Let $z_1 = \sqrt{2}(cos225^0 + isin225^0)$and $z_2 = \frac{1}{2}(\cos 150^0 + isin150^0)$, then find z_1z_2 and $\frac{z_1}{z_2}$
8. Compute the following

$a.\ (3(\cos 15^0 + isin15^0)^3$c. $(\cos 1^0 + i \sin 1^0)^{30}$ $b.\ (\sqrt{3} - i)^3$

9. Find the square root of 3- 2i
10. Find the fourth root of the number z = 1- i and z = i
11. Find the solution of each of the following

a. $z^5 = 32$ c. $(z-2)^3 = 125$

b. $z^4 = \sqrt{3} + i$ d. $z^6 = 1$

Unit Three
Further on Functions

3.1. Types of Functions and their Graphs

3.1.1 Polynomial Function

Introduction

In this section you will study the definition of polynomial function, basic operations on polynomial function, theorems on polynomial and the graphs of polynomial functions.

Definition of polynomial function

Definition: - A polynomial is an algebraic expression that can be written in the form of

$a_n x^n + a_{n-1}x^{n-1} + \cdots + \mathrm{a}_2 x^2 + \mathrm{a}_1\mathrm{x} + \mathrm{a}_0$,where$n, n-1, n-2, \ldots$,2,1,0 $are elements of the set of whole numbers$ $\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2$,…. a_n are elements of the set of real numbers, Degree"n" and x is variable.

Example: - Each of the following is a polynomial expression

a. 1,x , x^2 , x^3 , ….
b. $x + 1, x^2 + \mathrm{x} + 1, 3x^3\ \ 4x^2 + 5, \ldots$
c. $\sqrt{8}x^5 + 25x^4 + \pi x^2 + 0.5$

Note that:- Zero polynomial is the only polynomial which has no degree.

Types of Polynomials over $Z[x], Q[x], R[x]$

A polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, is said to be

1. A polynomial is over the set of integers if all the coefficients $a_0, a_1, a_2, \ldots.\ a_n \in Z$

   The set of polynomials over Z is denoted by $Z[x]$

2. A polynomial over the set of rational numbers if all the coefficient s $a_0, a_1, a_2, \ldots.\ a_n \in Q$

   The set of polynomials over Q is denoted by $Q[x]$

3. A polynomial over the set of real numbers if all the coefficients $a_0, a_1, a_2, \ldots.\ a_n \in R$

   The set of polynomial over R is denoted by $R[x]$

Example1:-

a. $x, x^3 + 4x - 1, 10x^{23} - 3x^2 + 11 \in Z[x] because 1 \in Z, 1,4,-1 \in Z, 10,-3,11 \in Z$

b. $\frac{1}{2}x, x^3 - 4x + \frac{1}{9}, x^7 - 13x + 0.71 \in Q[x] because \frac{1}{2} \in Q, 1,-4,\frac{1}{9} \in Q, 1,-13,0.71 \in Q$

c. $\pi^2, \pi x^2 + \sqrt{2}, \frac{1}{3}x - 7 \in R[x] because \pi \in R, \pi, \sqrt{2} \in R, \frac{1}{3}, -7 \in R$

Note: $Z[x] \subset Q[x] \subset R[x]$

Definition:-Let n be anon-negative integers and $a_n, a_{n-1}, \ldots \ldots.., a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. *The* function $p(x) = a_n x^n + a_{n-1}x^{n-1} +, \ldots, a_1 x +$ a_1 *is called apolynomial function in variable x of* degree n.

- ✓ **Note that**: In the definition of polynomial function $p(x) = a_n x^n + a_{n-1}x^{n-1} +, \ldots, a_1 x + a_0$

I. $a_n, a_{n-1}, \ldots \ldots.., a_2, a_1, a_0$ are the coefficient of polynomial function and which are an element of the set of real numbers . The number $a_n$is called the leading coefficient of the polynomial function and$a_n x^n$ is the leading term. The number a_0 is called the constant term of the polynomial.

II. n, n-1, n-2, …, 2, 1, 0 are the exponents of the polynomial function and which are elements of the set of whole numbers (non –negative integers). The number n (exponent of the highest power of x) is the degree of the polynomial.

Note that:- the domain of any polynomial function is the set of all real numbers.

Example: - which one of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term

a. $f(x) = 2x^2 - x + 7$ d. $f(x) = \frac{x^2+4}{x^2+4}$

b. $f(x) = \frac{x}{\pi}$ e. $f(x) = 2x^{-4} - x^2 + 8x + 17$

c. $g(x) = \frac{x}{x}$ f. $f(x) = \sqrt{(x+1)^2}$

Solution

a. It is a polynomial function of degree 2 with leading coefficient 2, leading term $2x^2$ and constant term 7
b. It is a polynomial function of degree 1 with leading coefficient $\frac{1}{\pi}$, leading term $\frac{x}{\pi}$ and constant term 0.
c. It is not a polynomial function because one of its domain is not real number
d. $f(x) = \frac{x^2+4}{x^2+4}$ =1, so it is a polynomial function of degree 0 with leading coefficient 1, leading term 1 and constant term 1.
e. It is not a polynomial function because one of its terms has negative exponent.
f. $f(x) = \sqrt{(x+1)^2}$ $=|x+1|$, so it is not polynomial because it cannot written in the form of $\mathrm{p}(x) = a_n x^n + a_{n-1}x^{n-1} +, \dots, a_1 x + a_0$

Activity 3.1

In each of the following polynomials list the leading coefficient, the degree and constant term .

a. $5x^2 + 6x + 7d. x^{100} - 5$
b. $x^6 - x^7 - x^2 + 2x$ e. $(2x+1)^3$
c. 0 f. 10

3.1.1.1 Basic operations on polynomial function

A. Addition of polynomials

You can add polynomial functions in the same way as you add real numbers simply add like terms by adding the like terms adding their coefficients.

Example: Let, $f(x) = x^3 + x^4 + 6x + 7$ $g(x) = x^3 + 2x^2 - 4x1$, $then\ find\ f(x) + g(x)$

Solution

$f(x) + g(x) = (x^3 + x^4 + 6x + 7) + (x^3 + 2x^2 - 4x - 1)$

$= x^4 + x^3 + x^3 + 2x^2 + 6x - 4x + 7 - 1$

$= x^4 + 2x^3 + 2x^2 - 4x + 6$

B. Subtraction of polynomials

If f(x)and g(x) are two polynomials then the difference off(x)and g(x)is found by subtracting the coefficient of the similar terms that is denoted by f(x)-g(x)

Example: - Let $f(x) = 2x^3 - 5x^2 + x - 7 \;\; g(x) = x^4 - x^3 + 5x^2 + 6x \;\; , then$

$find\ f(x) - g(x)$

Solution:-$f(x) - g(x) = (2x^3 - 5x^2 + x - 7) - (x^4 - x^3 + 5x^2 + 6x \;\;)$

$$= 2x^3 - 5x^2 + x - 7 - x^4 + x^3 - 5x^2 - 6x$$

$$= -x^4 + 2x^3 + x^3 - 5x^2 + x - 6x - 7$$

$$= -x^4 + 3x^3 - 10x^2 - 5x - 7$$

The difference of two polynomial functions $f\ and\ g\ is\ written\ as\ f - g\ and\ defined\ as$

$$(f - g)\ such\ that\ (f - g)(x) = f(x) - g(x) for\ all\ x \in R$$

- ✓ **Note that:-** If the degree of$f$ is not equal to the degree of g , then the degree of

$(f - g)(x)$is the degree of $f(x)$ or the degree $g)(x)$ whichever has the highest

degree. If they have the same degree, however, the degree of $(f - g)(x)$might be lower

than this common degree when they have the same leading coefficient.

Example:-- Let $f(x) = x^4 + 3x^3 - x^2 + 4\ and\ g(x)x^4 - x^3 + 5x^2 + 6x, then\ find\ the\ degree\ of\ (f - g)(x)$

Solution: $\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}) = (x^4 + 3x^3 - x^2 + 4) - (x^4 - x^3 + 5x^2 + 6x\ )$

$$= x^4 + 3x^3 - x^2 + 4 - x^4 + x^3 - 5x^2 - 6x$$

$$= x^4 - x^4 + 3x^3 + x^3 - x^2 - 5x^2 - 6x + 4$$

$= 4x^3 - 6x^2 - 6x + 4$This is a polynomial function of degree 3

C. Multiplication of polynomials

- If $f(x)\ and\ g(x) are\ two\ polynomials\ then\ the\ product\ of\ f(x)\ and\ g(x) is\ denoted\ by\ f(x).g(x)$. To multiply two polynomial functions, multiply each term of one by each term of others.

Example:- $Let\ f(x) = 2x^3 - 2x^2 + 3x - 2\ and\ g(x) = x^3 - 2x + 1$, then $find\ f(x)g(x)$.

Solution:-$f(x)g(x) = (2x^3 - 2x^2 + 3x - 2)(x^3 - 2x + 1\ )$

$$= 2x^3(x^3 - 2x + 1\ ) - 2x^2(x^3 - 2x + 1\ ) + 3x(x^3 - 2x + 1\ ) - 2(x^3 - 2x + 1\ )$$

$$= 2x^6 - 4x^4 + 2x^3 - x^5 + 2x^3 2x^3 - x^2 + 3x^4 - 6x^2 + 3x - 2x^3 + 4x - 2$$

$$= 2x^6 - x^5 - 4x^4 + 3x^4 + 2x^3 + 2x^3 - 2x^3 - x^2 - 6x^2 + 3x + 4x - 2$$

$$= 2x^6 - x^5 - x^4 + 2x^3 - x^2 + 7x - 2$$

The product of two polynomial functions f and g is written as $f.g$ and define as

$$(f.g)(x) = f(x).g(x) for\ all\ x \in R$$

- ✓ $\boldsymbol{Note\ that}: -$ $The\ degree\ of\ the\ product\ of\ two\ polynomials\ f(x)\ and\ g(x)$

  $is\ the\ sum\ of\ the\ degreeof\ f(x) and\ g(x)\ where\ f(x) \neq 0\ and\ g(x) \neq 0.$

D. Division of Polynomials

- In dividing a polynomials one by the we use long division and we should continue until the remainder is zero polynomial or a polynomial of lower degree than the of divisor.
- Suppose we have two polynomials$N(x)$of degree n and $D(x)$ of degree m, where $n \geq m$ when we divide $N(x)\ by\ D(x)$ we obtained a quotient $Q(x)$ and a remainder $R(x)$.
  i.e$Dividend = (divisor)(quotient) + remainder$

$$N(x) = D(x)Q(x) + R(x)$$

Long Division Method

When you are asked to divide one polynomial by another, stop the division process when you get a quotient and remainder that polynomial and degree of the remainder is less than the degree of the divisor

Example 1:- Divide $x^2 + 3x + 1\ by\ x + 1$

Solution:-

In this division of polynomial $x^2 + 3x + 1\ i$s called dividend and the polynomial

$x + 2$ ← Quotient

Devisor → $x + 1$ | $x^2 + 3x + 1$ ← Devaidend

- $(x^2 + x)$

$2x + 1$

-$(2x + 2)$

-1 ← Remainder

$$N(x) = D(x)Q(x) + R(x)$$

$$x^2 + 3x + 1 = (x + 1)(x + 2) - 1$$

Example: - Divide $4x^4 - x^2 + 2x + 1 \; by \; x^2 + 2x + 1$

Solution:-

$$4x^2 - 8x + 1 \quad \longleftarrow \text{Quotient}$$

$$\text{Divisor} \longrightarrow x^2 + 2x + 1 \;\big|\; 4x^4 - x^2 + 2x + 1 \quad \longleftarrow \text{Dividend}$$

$$-(4x^4 + 8\,x^3 + 4x^2)$$

$$-8x^3 - 5x^2 + 2x + 1$$

$$-(-8x^3 - 16x^2 - 8x)$$

$$11x^2 + 10x + 1$$

$$-(11x^2 + 22x + 11)$$

$$-12x - 10 \quad \longleftarrow \text{Remainder}$$

Therefore $\mathrm{N(x) = D(x)Q(x) + R(x)}$

$$4x^4 - x^2 + 2x + 1 = (x^2 + 2x + 1)(4x^2 - 8x + 11) + (-12x - 10)$$

3.1.1.2. Theorems on polynomial function

1. Polynomial division theorem

Recall that when we divided one polynomial by another we apply the long division

Long division; - The division should continue until the remainder is either zero or degree of remainder less than the degree of the divisor.

Theorem 1.1:- **Polynomial division theorem**

f(x)andd(x)arepolynomialssuchthatd(x) $\neq$ 0, andthedegreeofd(x)islessthanor equal tothedegreeoff(x), thenthereexistuniquepolynon

f(x) = d(x)q(x) + r(x), wherer(x) = 0 orthedegreeofr(x)isless thanthedegreeofd(x). Iftheremainderr(x)iszero , f(x)dividesexactlyintod(x).

Example1:- Divide $x^3 - 2x^2 + x + 5 \; by \; x^2 - x + 3$

Solution

$$\begin{array}{r|l} & x-1 \quad \leftarrow \text{Quotient} \\ \hline \textit{Divisor} \longrightarrow x^2-x+3 & x^3-2x^2+x+5 \leftarrow \text{Dividend} \\ & -(x^3-x^2+3x) \\ \hline & -x^2-2x+5 \\ & -(-x^2+x-3) \\ \hline & -3x+8 \leftarrow \text{Remainder} \end{array}$$

<u>Check</u> $Dividend = (Divisor)(Quotient) + Remainder$

$x^3 - 2x^2 + x + 5 = (x^2 - x + 3)(x - 1) + (-3x + 8)$

Example 2:- In each of the following pairs of polynomials, find polynomials q(x) and r(x) such that $f(x) = d(x)q(x) + r(x)$

a. $f(x) = x^3 - x^2 + 5, d(x) = x + 1$
b. $f(x) = 2x^4 - x^2 + 1, \; d(x) = x^2 + x$

Solution:-

a. $\frac{f(x)}{d(x)} = \frac{x^3-x^2+5}{x+1} = (x + 1)(x^2 - 2x + 2) + (3)$

Therefore, $q(x) = x^2 - 2x + 2 \; and \; r(x) = 3$

b. $\frac{f(x)}{d(x)} = \frac{2x^4-x^2+1}{x^2+x} = (x^2 + x)(2x^4 - x^2 + 1) + (-x + 1)$

There fore $q(x) = 2x^2 - 2x + 1 \; and \; r(x) = -x + 1$

2.Theorem 1.2:- Remainder Theorem

> *Let $f(x)$ be a polynomial of degree greater than or equal to 1 and if the polynomial function $f(x)$ is divided by $x - c$ .then the remainder $R = f(c)$*

Proof:-

When $f(x)$ *is divided by* $x - c$ *, the remainder is always a constant . by the polynomial division theorem ,* $f(x) = (x - c)q(x) + k$ *, where k is constant .*

This equation holds for every real number x. hence , it holds when $x = c$

In particular, if you let $x = c$, observe a very interesting and useful relationship

$$f(c) = (c - c)q(c) + k$$

$$= 0.q(c) + k$$

$$= 0 + k = k$$

It follows that the value of the polynomial $f(x)$ $at\ x = c\ is\ the\ same\ as\ theremainder\ k$

$$obtained\ when\ you\ divided\ f(x) by\ x - c$$

Example: - Find the remainder when$f(x) = x^3 + 5x^2 - 11x + 7\ is\ divided\ by\ x - 2$

Solution:-$c = 2, f(2) = (2)^3 + 5(2)^2 - 11(2) + 7$

$$= 8 + 20 - 22 + 7$$

$$= 13$$

Example: - Find the remainder when $f(x) = 105x^{75} - 11x^2 + 9\ is\ divided\ by\ x + 1$

Solution:-$c = -1, f(-1) = 105(-1)^{75} - 11(-1)^2 + 9$

$$= -107$$

Example: - When $x^3 - 2x^2 + 3bx + 10\ is\ divided\ by\ x - 3\ the\ remainder\ is\ 37$.

$find\ the$ Value of b.

Solution:- Let $f(x) = x^3 - 2x^2 + 3bx + 10$

$$f(3) = 37 (by\ the\ remainder\ theorem)$$
$$\Rightarrow\ (3)^3 - 2(3)^2 + 3b(3) + 10 = 37$$

$$\Rightarrow 27 - 18 + 9b + 10 = 37$$

$$\Rightarrow 9b + 19 = 37$$

$$\Rightarrow b = 2$$

Exercise 3.1

1. Find the remainder in the following pairs of polynomials using polynomial divisions and the remainder theorem
    a. $f(x) = x^3 - 2x^2 + 8x - 1, q(x) = x - 2$
    b. $f(x) = x^{17} - 1, q(x) = x + 1$
    c. $f(x) = 2x^2 + 3x + 1, q(x) = 2x + 1$

2. When $f(x) = 3x^7 - a\,x^6 + 5\,x^3 - x + 11$ $is\ divided\ by\ x + 1\ , the\ remainder is 15.$ $what\ is\ the\ value\ of\ a?$
3. When the polynomial $f(x) = a\,x^3 + b\,x^2 - 2x + 8$ $is\ divided\ by\ x - 1\ and\ x + 1\ the remainders are\ 3 and\ 5\ respectively.\ find the value of a and b.$

3. Theorem 1.3 Factor Theorem

Let $f(x) be a polynomial of degree greater than or equal to one and let c be any real\ number, then$

i. $x - c is a factor of f(x), if f(c) = 0\ , and$

ii. $f(x) = 0\ , if x - c is a factor of f(x)$

Example1: - Show that $x - 2\ is a factor of x^3 - 2\,x^2 + 5x - 10$

Solution:-$c = 2, f(2) = (2)^3 - 2\,(2)^2 + 5(2) - 10$

$$= 8 - 8 + 10 - 10$$

$$= 0$$

$$since\ , f(2) = 0, then x - 2\ is a factor of x^3 - 2\,x^2 + 5x - 10$$

Example2:- Show that $x + 3\ , x - 2 and x + 1\ are factors and x + 2\ is not factor of$

$$f(x) = x^4 + x^3 - 7x^2 - x + 6$$

Solution:- $f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6$

$$= 8 - 8 + 10 - 10$$

$$= 0$$

$$Hence, x + 3\ is a factor of f(x)$$

$$f(2) = (2)^4 + (2)^3 - 7(2)^2 - (2) + 6$$

$$= 16 + 8 - 28 - 2 + 6$$

$$= 0$$

$$Hence, x - 2\ is factor of f(x)$$

$$f(-1) = (-1)^4 + (-1)^3 - 7(-1)^2 - (-1) + 6$$

$$= 1 - 1 - 7 + 1 + 6$$

$$= 0$$

$$Hence, x + 1\ is a factor of f(x)$$

$$f(-2) = (-2)^4 + (-2)^3 - 7(-2)^2 - (-2) + 6$$

$$= 16 - 8 - 28 + 2 + 6$$

$$= -12$$

$$Hence, x + 2\ isnofactoroff(x)$$

Activity 3.2

1. In each of the following , use the factor theorem to determine whether or not g(x) is a factor of f(x)
   a. $g(x) = x + 1, f(x) = x^5 + 4x^3 + 12$
   b. $g(x) = x + 2, f(x) = x^3 - 3x^2 - 4x - 12$
   c. $g(x) = x + 1, f(x) = x^{25} + 1$
2. Find the value of k if $x + 3\ isafactorofx^5 - kx^4 - 6x^3 + x^2 + 4x + 29$

3.1.1.3. Zeros of polynomial functions

A real number c is said to be a zero of function $fiff(c) = 0$

According to the degree of the polynomial equations can be categorized as

i. Linear equations(first degree polynomial equations)
ii. Quadratic equations (second degree polynomial equations)
iii. Higher degree polynomial equations

i. First degree polynomial (linear)equations

Linear equation is an equation of degree one which can be reduced to an equation of the $ax + b = 0, wherea \neq 0, a, b\epsilon R$

For example $2x - 3 = 0, 1 - 2x = 3, etc. arelinearequations.$ solving linear equation means finding the zeros of the given linear equation.

Example1: - Find the zeros

a. $f(x) = 2x + 1$
b. $f(x) = 3 - (2x + 4) + x + 5$

Solution :-

a. $f(x) = 0$

$$\Rightarrow 2x + 1 = 0$$

$$\Rightarrow 2x = -1$$

$$\Rightarrow x = -\frac{1}{2}$$

b. $f(x) = 0$

$$\Rightarrow 3 - (2x + 4) + x + 5 = 0$$

$$\Rightarrow 3 - 2x - 4 + x + 5 = 0$$

$$\Rightarrow 4 - x = 0$$

$$\Rightarrow x = 4$$

$$f(4) = 3 - (2(4) + 4) + (4) + 5$$

$$= 0$$

$$The zero is\ 4$$

ii. Quadratic equation

Quadratic equation is an equation of degree two which can be reduced to an equation of the form $ax^2 + bx + c = 0, for some fixed real numbers a, b and c with a \neq 0$

Example1: - Find the zeros of the following function

a. $f(x) = x^2 - 7x + 6$
b. $f(x) = x^2 - 36$

Solution :-

a. $f(x) = 0$

$x^2 - 7x + 6x^2 - 7x + 6 = 0$

$\Rightarrow x^2 - x - 6x + 6 = 0$

$\Rightarrow x(x - 1) - 6(x - 1)$

$\Rightarrow (x - 6)(x - 1) = 0$

$\Rightarrow x = 6, or x = 1$

$f(6) = (6)^2 - 7(6) + 6$

$\Rightarrow 36 - 42 + 6 = 0$

$f(1) = (1)^2 - 7(1) + 6$

$= 0$

$Therefore, 1 and\ 6\ are the zeros of f.$

b. $f(x) = 0$

$$\Rightarrow x^2 - 36 = 0$$

$$\Rightarrow x = 6, or x = -6$$

$$f(6) = (6)^2 - 36$$
$$= 0$$

$$f(-6) = (-6)^2 - 36$$
$$= 0$$

$Therefore, -6\ and\ 6\ are the zeros of f$

Note: -A polynomial function cannot have more zeros than its degree.

iii. Higher degree polynomials

There is no simple general way in which a root can be determined exactly when the degree of polynomial is $n > 2$. in deed, the location theorem will be applied to find the approximate value of where the graph of p(x) crosses the x-axis .

A real root of$p(x) = 0\ or a zeros of p(x) corresponds to values of x at which$

$$the graph of y = p(x) crosses or touches the e x - axis.$$

Rational root test

The rational root test relates the possible rational zeros of apolynomial with integer coefficients to the leading coefficient and to the constant term of the polynomial.

4. **Theorem1.4Rational root test**.

If the rational number $\frac{p}{q}, in its lowest terms is a zero of the polynomial$ $f(x) = a_n x^n + a_{n-1}x^{n-1} + --- + a_1 x + a_0$with integer coefficients then **p** must be a factor of $\mathbf{a_0}$and **q** must be a factor of $\mathbf{a_n}$.

Example1: - In each of the following find all the rational zeros of the polynomials.

a. $f(x) = 2x^3 + 9x^2 + 7x - 6$
b. $f(x) = \frac{1}{2}x^4 - 2x^3 - \frac{1}{2}x^2 + 2x$
c. $h(x) = x^3 - x + 1$

Solution:-

$$a.\ f(x) = 2x^3 + 9x^2 + 7x - 6$$

$a_n = a_3 = 2\ and\ a_0 = -6$

possible value of p are factors of -6 . these are $\pm 1, \pm 2, \pm 3\ and \pm 6$

possible value of q are factors of 2. These are $\pm 1\ and \pm 2$

$$possible rational zeros \frac{p}{q} are \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2} and \pm \frac{3}{2} of these\ 12\ possible$$

$$rational\ zeros\ at\ most 3\ can\ be\ zeros\ of\ f$$

$f(-1) = 2(-1)^3 + 9(-1)^2 + 7(-1) - 6$

$= -2 + 9 - 7 - 6$
$= -6 \neq 0$

$$f(1) = 2(1)^3 + 9(1)^2 + 7(1) - 6$$

$= -2 + 9 + 7 - 6$
$= 12 \neq 0$

$$f(-2) = 2(-2)^3 + 9(-2)^2 + 7(-2) - 6$$

$= -16 + 36 - 14 - 6$
$= -36 + 36$
$= 0$

Similarly;

$$f(2) \neq 0, \quad f(3) \neq 0, f(-6) \neq 0, \quad f(6) \neq 0, f\left(-\frac{1}{2}\right) \neq 0, f\left(\frac{1}{2}\right) = 0$$

Using the factor theorem, we can factorize

$f(x)\ as\ \ 2x^3 + 9x^2 + 7x - 6 = (x + 3)(x + 2)\left(x - \frac{1}{2}\right)$, So

$$g(x) = 0\ at\ x = -3, x = -2, and\ x = \frac{1}{2}$$

Therefore, $-3, -2 and \frac{1}{2}$ $are\ the\ only\ (rational) zeros\ of\ f$

$b.\ \ g(x) = \frac{1}{2}\ x^4 - \ 2x^3 - \frac{1}{2}\ x^2 + 2x$

Multiply both sides by 2 to make coefficient is an integer
$2. g(x = \ x^4 - \ 4x^3 - \ x^2 + 4x$
Let $h(x) = 2g(x)$
$\Rightarrow h(x) = \ x^4 - \ 4x^3 - \ x^2 + 4x$
$h(x) = x(x^3 - 4x^2 - x + 4) = xk(x)$
$k(x) = \ x^3 - 4x^2 - x + 4$

k(x) has leading coefficient of 1 and constant 4

The possible value of p are factors of 4 . thse are $\pm 1, \pm 2$ and ± 4

The possible value of q are factors of 1. These are ± 1

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2$ and ± 4

$$k(-1) = (-1)^3 - 4(-1)^2 - (-1) + 4$$

$$= -1 - 4 + 1 + 4 = 0$$

$$k(1) = (1)^3 - 4(1)^2 - (1) + 4$$
$$= 1 - 4 - 1 + 4$$
$$= 0$$

$k(-2) \neq 0$
$k(2) \neq 0$
$k(-4) \neq 0$
$k(4) \neq 0$

So by factor theorem $k(x) = (x + 1)(x - 1)(x - 4)$

Hence, $h(x) = xk(x) = x(x + 1)(x - 1)(x - 4)$

$h(x) = 2g(x)$
$h(x) = 2x(x + 1)(x - 1)(x - 4)$
Therefore, the zeros of g(x) are $0, \pm 1$ and 4

d. $a_n = 1, a_0 = 1$

The possible rational zeros are ± 1. Using the remainder theorem ,test these possible roots (zeros)

$f(1) \neq 0$

$f(-1) \neq 0$

So, we can conclude that the given polynomial has no rational zeros.

Activity 3.3

1. In each of the following, find all the rational roots of the polynomial.
   a. $f(x) = x^3 + 3x^2 - x - 3$
   b. $f(x) = x^4 + 3x^3 - 11x^2 - 3x + 10$
   c. $g(x) = x^4 - x^3 - x^2 - x - 2$
   d. $g(x) = 2x^3 + 5x^2 - 11x + 4$
2. In each of the following find all real solutions of the equations.
   a. $2x^3 - 5x^2 + x + 2 = 0$
   b. $x^3 + x^2 + 2x + 2 = 0$
   c. $x^4 - x^3 - x^2 - x - 2 = 0$

Zeros and their multiplicities

If $(x - c)^m$ is afactor of $f(x)$, but $(x - c)^{m+1}$ is not a factor of $f(x)$, then c is said tobe a zero of multiplicity m.

Example1: - Let $f(x) = x^{75}(x - 1)^3(x - \pi)^9$,

Solution:-

The zeros of $f(x)$ are 0,1 and π with multiplicity 75, 3 and 9 respectively.

Example2: - List zeros with their multiplicity for the polynomial

$$f(x) = (x^2 - 4)^5 (x^2 - x - 2)^2(x^2 - 7)^3$$

Solution:-$(x^2 - 4)^5 = \left((x + 2)(x - 2)\right)^5 = (x + 2)^5 (x - 2)^5$

$$(x^2 - x - 2)^2 = \left((x - 2)(x + 1)\right)^2 = (x - 2)^2(x + 1)^2$$

$$(x^2 - 7)^3 = \left(\left(x - \sqrt{7}\right)\left(x + \sqrt{7}\right)\right)^3 = \left(x - \sqrt{7}\right)^3\left(x + \sqrt{7}\right)^3$$

$\Rightarrow f(x) = (x^2 - 4)^5 (x^2 - x - 2)^2(x^2 - 7)^3$

$$= (x + 2)^5 (x - 2)^5 (x - 2)^2(x + 1)^2\left(x - \sqrt{7}\right)^3\left(x + \sqrt{7}\right)^3$$

$$= (x - 2)^7 (x + 2)^5 (x + 1)^2\left(x - \sqrt{7}\right)^3\left(x + \sqrt{7}\right)^3$$

zeros	2	-2	-1	$\sqrt{7}$	$-\sqrt{7}$
multiplicity	7	5	2	3	3

3.1.1.4. Graphs of polynomial functions

1. Graphs of linear functions

If a and b are fixed real numbers a≠0 , then f(x)=ax+b for x∈R is called a linear function .

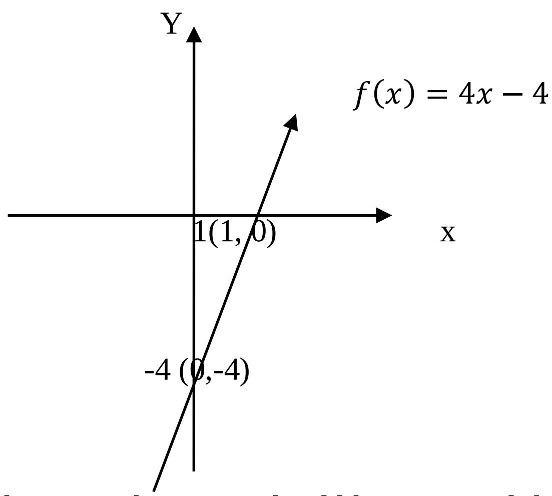
If a=0 , then f(x)=b is called constant function .some time linear functions are written as y=ax+b

Example1: - Draw the graph of the linear function. $f(x) = 4x - 4$

Solution:-a. First you construct a table of values from the domain

x	-4	-3	-2	-1	0	1	2	3	4
f(x)	-20	-16	-12	-8	-4	0	4	8	12

b. Now you plot these points on a coordinate system and draw a line through these points



From the graphs given above you should have noticed that

1. Graphs of linear functions are straight lines
2. If $a > 0$, *then the graph of the linear function* $f(x) = ax + b$ is increasing ,
3. If $a < 0$, *then the graph of the linear function* $f(x) = ax + b$ is decreasing
4. If $a = 0$, then the graph of the constant function $f(x) = b$ is horizontal line
5. if $x = 0$, then $f(0) = b$. This means $(0, b)$ lies on the graph of the function and the graph passes through the ordered pair $(0, b)$. This point is called the the $y -$ intercept . It is point at which the graph intersecct the $y -$ axis.
6. If $f(x) = 0$, then $0 = ax + b \Rightarrow x = \frac{-b}{a}$. This means $\left(\frac{-b}{a}, 0\right)$ lies on the graph on the function and the graph passes through the ordered pair $(-\frac{-b}{a}, 0)$.

   This point is called the $x -$ intercept . it is the point at which the graph intersects the $x -$ axis .

2.Graphs of quadratic function

Let $f(x) = ax^2 + bx + c, a \neq 0$ be a quadratic function

i. The graph of a quadratic function is a parabola
   - If $a > 0$, *the parabola opens upward*
   - If $a < 0$, *the parabola opens down ward*

ii. Vertex $(\frac{-b}{2a}, f(\frac{-b}{2a})) = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$ is the vertex of the parabola

- If $a > 0, f\left(\frac{-b}{2a}\right) = \frac{4ac-b^2}{4a}$ is minimum value of f(x) and its range is $\left[\frac{4ac-b^2}{4a}, \infty\right)$
- If a$< 0, f\left(\frac{-b}{2a}\right)$ is the maximum value of f(x) and its range is $\left(-\infty, \frac{4ac-b^2}{4a}\right]$

iii. $f(0) = c \Rightarrow (0, c)$ is the y − intercept

iv. If $b^2 - 4ac \geq 0$, then the graph cross the axis at $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- It meet the x − axis at exactly one point $x = \frac{-b}{2a}$.
- At two distinct points if $b^2 - 4ac > 0$
- It does not touch the x-axis if $b^2 - 4ac < 0$

Example1: - Draw the graph of $f(x) = 3x^2 + 11x - 4$

Solution: $-f(x) = 3x^2 + 11x - 4$

i. a> 0, *that means* $a = 3 > 0$, *the parabola opens upward*

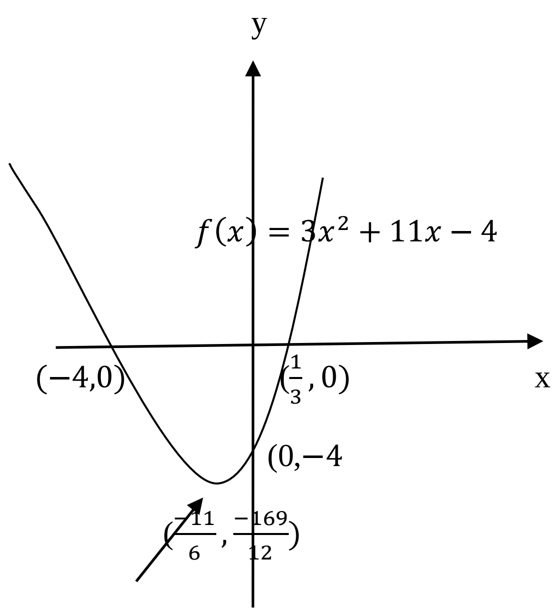
ii. vertex $=\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right) = \left(\frac{-11}{6}, \frac{4X3(-4)-(11)^2}{4(3)}\right) = \left(\frac{-11}{6}, \frac{-169}{12}\right)$

iii. $f(0) = -4$, the graph as y − intercept the point $(0, -4)$

iv. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = x = \frac{-11 \pm \sqrt{169}}{6}$

$$\Rightarrow x = \frac{1}{3} \text{ or } x = -4 \text{ this means } \left(\frac{1}{3}, 0\right) \text{and} (-4,0) \text{ are } x - \text{intercepts}$$

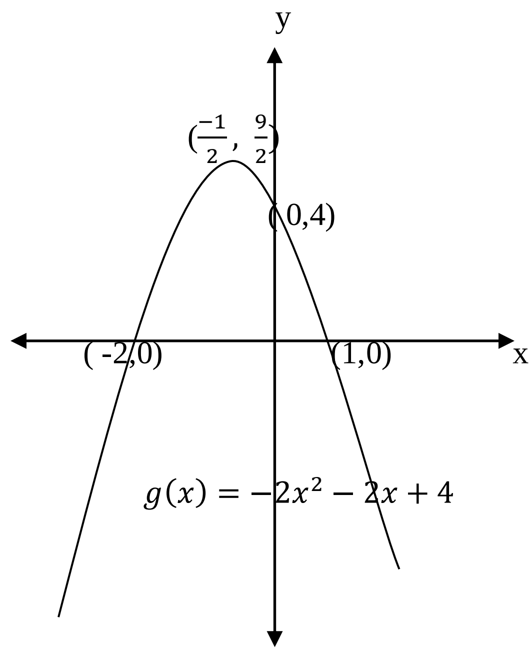
v. The minimum value of $y = \frac{-169}{12}$ and range= $\left[\frac{-169}{12}, \infty\right)$



Example: - Sketch the graph of $g(x) = -2x^2 - 2x + 4$

Solution;

i. The parabola opens downward
ii. Vertex $(\frac{-1}{2}, \frac{9}{2})$
iii. The range $\left(-\infty, \frac{9}{2}\right)$
iv. $\text{y} - \text{intercept } (0,4)$ and $\text{x} - \text{intercept } (-2,0) \text{and} (1,0)$
v. The function g has maximum value and it occurs at x = $\frac{-1}{2}$ and the maximum value is $f\left(\frac{-1}{2}\right) = \frac{9}{2}$



3. The graph of any polynomial function

In general the graph of every polynomial function of the form

$p(x) = a_n x^n + a_{n-1} x^{n-1} +, ..., a_1 x + a_0$ has the following properties

i. The graph of polynomial function of degree n has at most n-1 vertex or turning point.
ii. The domain of a polynomial function is the set of all real numbers and the range is the sub set real numbers or it could be the whole set
iii. The graph of polynomial function crosses the y-axis once and it crosses x-axis at most n –times (for polynomial degree n)
iv. The graph of any polynomial is a continuous curve with no hole or no sharp corners.
v. If the degree of the polynomial is odd, then the graph cross the x-axis at once because one direction of the graph goes downward and the other direction of the graph goes upward without end.

vi. If the degree of polynomial even ,then the graph may or may not cross the x-axis ,because both direction of the graph goes up ward or goes downward above or below the x-axis depending on the sign of leading coefficient .

Example1: - Sketch the graph of the following polynomial function.

a. $f(x) = x^3 - x^2 - x + 1$ $\quad b. f(x) = x^4 - 4x^2 + 4$

Solution:-

a. $f(x) = x^3 - x^2 - x + 1$

To find x-intercepts put $f(x) = 0,$

$$\Rightarrow x^3 - x^2 - x + 1 = 0, factors\ of\ 1\ are \pm 1$$

$$\Rightarrow f(-1) = 0,$$

$Hence, x + 1\ is\ factorof\ f(x) = x^3 - x^2 - x + 1$

$$\Rightarrow f(1) = 0,$$

$Hence\ , x - 1\ is\ factorof\ f(x) = x^3 - x^2 - x + 1$

$$f(x) = (x-1)(x-1)(x+1) = 0 \Rightarrow x = 1\ and\ x = -1$$

$$Thus, -1\ and\ 1\ arex - intercepts$$

i. To find y-intercept, put $x = 0 \Rightarrow f(0) = 1\ is\ y - intercept$
ii. Turning point

$f(x) = x^3 - x^2 - x + 1$ has degree 3 and therefore has at most a total 2 relative maxima and minima the exact location of turning points ,however requires higher mathematics but we show the shape of the graph by using x- intercept divide x-axis in to intervals.

$Thus, f(x) = x^3 - x^2 - x + 1\ we\ first\ factorize$

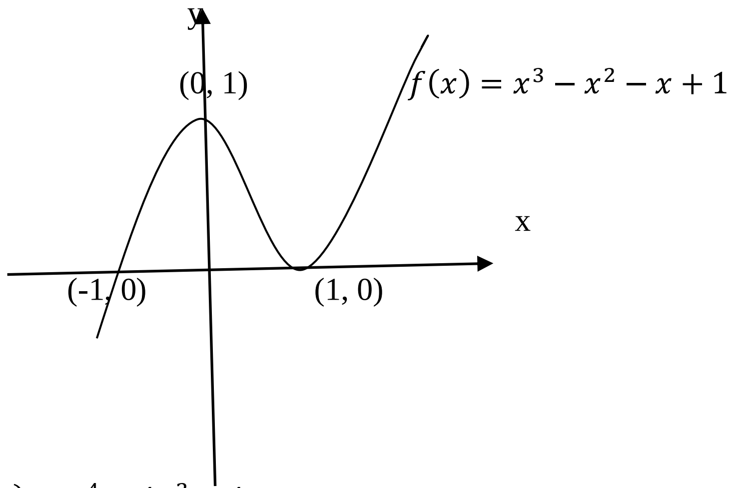
$$f(x) = (x-1)^2(x+1), the\ roots\ are\ x = -1\ and\ x = 1\ .$$

Now sketch sign chart

		-1		1	
$x+1$	$-$	0	$+ \quad +$		$+$
$(x-1)^2$	$+$		$+ \quad +$	0	$+$
$(x+1)(x-1)^2$	$-$	**0**	$+ \quad +$	**0**	$+$

Graph of $f(x)$ $\underbrace{below\ x-axis}$ $\underbrace{above\ x-axis}$ $\underbrace{above\ x-axis}$

Hence, the graph based on the above information will be below



b. $f(x) = x^4 - 4x^2 + 4$

i. To find x-intercepts put $y = f(x) = 0$,

$$\Rightarrow x^4 - 4x^2 + 4 = 0$$

Let $u = x^2$ then $x^4 - 4x^2 + 4 = 0$,

$$\Rightarrow u^2 - 4u + 4 = 0$$

$$\Rightarrow (u-2)^2 = 0, \Rightarrow u = 2$$

$$\text{Thus } x^2 = u, \Rightarrow x^2 = 2$$

$\Rightarrow x = \pm\sqrt{2}$, that are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are coordinatepoints ofthe x − intercepts

ii. To find y − intercept, put $x = 0$, $f(0) = 4$ is y − intercept $(0,4)$ is the coordinate point of y − intercepts

iii. Turning point

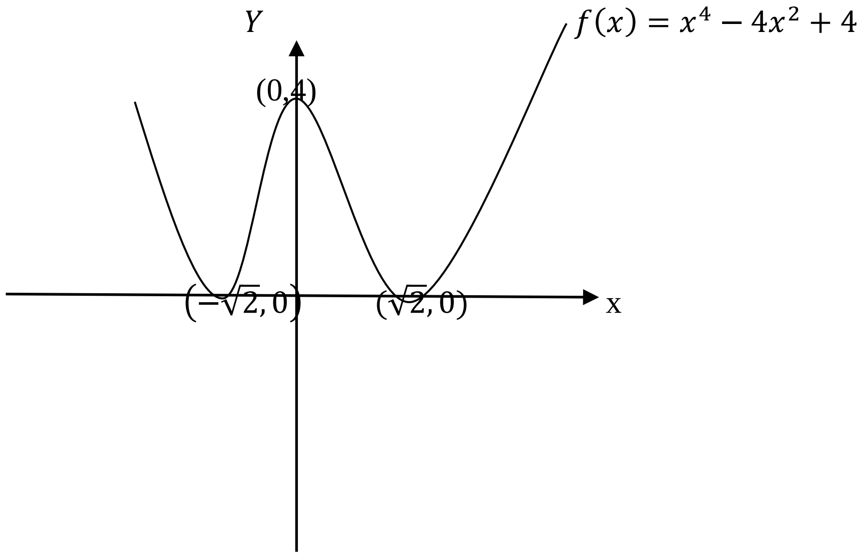
$f(x) = x^4 - 4x^2 + 4$ has degree 4 and at most a total of 3 relative maxima and minima . We show the shape of graph by using x-intercepts divide the x-axis into intervals. Thus $f(x) = x^4 - 4x^2 + 4$ first factorize

$= (x + \sqrt{2})^2 (x - \sqrt{2})^2$, the roots of f(x) are $x = -\sqrt{2}$ and $x = \sqrt{2}$,

hencef(x)has no roots in any of these intervals $(-\infty, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, \infty)$

Now sketch sign chart

	$x<-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}<x<\sqrt{2}$	$\sqrt{2}$	$x>\sqrt{2}$
$(x-\sqrt{2})^2$	+		+	0	+
$(x+\sqrt{2})^2$	+	0	+		+
$(x-\sqrt{2})^2(x+\sqrt{2})^2$	+	0	+	0	+
Graph of f(x)	Above x-axis		Above x-axis		Above x-axis



3.1.2. Rational function

Introduction:- In this section we defined a rational expression and determine the domain of a rational function , will show how to add, how to subtract, how to multiply, and how to divide one rational function by other and finally we will show how to simplify a given rational function.

Definition of Rational function

Definition 1:- A rational expression is the quotient $\frac{N(x)}{D(x)}$ of two polynomials N(x) and D(x) is not zero polynomial.
Here, N(x) is called Numerator and D(x) is called Denominator and D(x)$\neq$ 0.

Example 1 :- a) . $\frac{x^3+3x-1}{x-1}$ is **a rational expression**

b/c the numerator $x^3 + 3x - 1$ is polynomialof degree 3 and the

denominator $x - 1$ is a polynomial of degree 1.

b) $\frac{-x^3+2x+1}{\sqrt{x}+1}$is **not a rational expression**

b/c the denominator $\sqrt{x} + 1$ is not polynomial.

c) 7 is **a rational expression**

b/c 7 is a constant polynomial.

Note that:- i. Every polynomial is a rational expression.

ii. Every rational is not a polynomial expression.

Activity 3.5

- Determine whether the following expression defined rational expression or not.

a) $\frac{x^2+1}{\sqrt{2-x}}$ b) $\frac{x}{x}$c)$\frac{\sqrt{x^2}}{x^2+x-2}$ d)$\frac{x^2-5x+1}{x^{-2}-2x}$ e)$\frac{x^4+3x-5}{3^x}$

f)$\frac{(x-1)(x+1)}{x}$g)$-x^{-3} + 2x^2 + 1$ h) $\sqrt{\frac{x-2}{x+2}}$ i)$\frac{x+2}{\sqrt{x^2}}$ j) $\frac{\sqrt[3]{xy}}{xy+1}$

Definition 2:-
A rational function of x is a function R which is defined by $R(x) = \frac{N(x)}{D(x)}$,where$N(x)$ *and* $D(x)$ are polynomials and $D(x)$ is not zero polynomial ($D(x) \neq 0$) is called a Rational Function.

Example 1.$a. f(x) = \frac{1}{x+2}$ $b.\ r(x) = x - 2$

$c.\ h(x) = \frac{x^3+2x^2-8}{x^2-8}$ $d.\ g(x) = \frac{x+2}{(x-1)(x+1)}$ **,** all are rational functions**.**

N. B: - Any polynomial function P(x) is a rational function B/c we can written $p(x) = \frac{p(x)}{1}$, where$p(x) = 1$ is a polynomial function.

Activity 3.6

- Determine which of the following are rational functions?

$a)\ g(x) = \frac{x-2}{2x^3+x^2-2}$b). $f(x) = x^2 + 3x + 5$ $c).\ k(x) = \sqrt{9 - x^2}$

Domain and Range of Rational Function

Definition 3:- The domain of the rational function $F(x) = \frac{N(x)}{D(x)}$ consists of all real numbers except for all zeros of D(x). i.e $D(x) \neq 0$.
The domain of F can be denoted by $D_f = \{x: x \in \Re \ and \ \ D(x) \neq 0\}$.

Example 1:-Let $(\boldsymbol{x}) = \frac{\boldsymbol{x+2}}{(\boldsymbol{x+1})(\boldsymbol{x-1})}$**,** then

Domain of $\boldsymbol{R}(\boldsymbol{x}) = \{x: (x+1)(x-1) \neq 0\}$

$$= \{x: (x+1) \neq 0 \ and \ (x-1) \neq 0\}$$

$$= \{x: x \neq -1 \ and \ x \neq 1\}$$

i.e. Domain of R(x) is the set of all real number except – 1 and 1.

It denoted by Domain of $R(x) = \Re/\{-1,1\}$

Example 2:- Let$f(x) = \frac{x-1}{x^2+2}$, then the domain of $f(x) = \{x: x^2 + 3 \neq 0\}$.But, $x^2 + 3$ can not be zero in the set of real numbers, hence, the domain of $f(x)$ is the set of all real numbers and it denoted by Domain of $R(x) = \{x: x \in \Re\}$

Example 3:-let $(x) = \frac{x}{2x^2-7x+3}$, then

Domain of $g(x) = \{x: 2x^2 - 7x + 3 \neq 0\}$

$$= \{x: (2x-1)(x-3) \neq 0\}$$

$$= \{x: (2x-1) \neq 0 \ and \ (x-3) \neq 0\}$$

$$= \{x: 2x \neq 1 \ and \ x \neq 3\}$$

$$= \left\{x: x \neq \frac{1}{2} \ and \ x \neq 3\right\}$$

i.e Domain of $\boldsymbol{g}(\boldsymbol{x})$is the set of all real numbers except $\frac{1}{2} and$ 3.

It denoted by Domain of $g(x) = \Re/\left\{\frac{1}{2}, 3\right\}$

Activity 3.7

- Find the domain of each of the following rational functions.

a) $f(x) = \frac{x-1}{3x+2}$ b) $g(x) = \frac{x^2+3x-2-1}{x^2+1}$ c) $h(x) = \frac{1}{x^2+x-2}$

d)$l(x) = \frac{3-x}{x^2-9}$e) $m(x) = x^{-2}$ f)$n(x) = \frac{x^2+1}{x^3-4x}$

✓ **Self test exercise 3.1**

1. Which of the following is rational expiration?

   a. $2+\frac{1}{2}x^2$ b. $\frac{-x^4+3x-5}{2x}$ c. $\frac{-a^2}{3a-1}$ d. $\sqrt{\frac{x-2}{x+2}}$ e. $log_{\frac{1}{4}}64$

   f. $\frac{x}{4\pi}$ g. $\frac{x^2+2x+1}{x^2-x-2}$ h. $\frac{15\sqrt{3}}{104}$ i. $\frac{1-x^2}{\sqrt{x^2+1}}$

2. Determine the domain of each rational function

a. $k(x)=5+\frac{5}{x}$ b. $h(x)=\frac{x+2}{(x+2)(x-3)}$ c. $f(x)=\frac{x^3-27}{24}$

d. $j(x)=\frac{-5}{x^4}$ e. $p(x)=1+\frac{x^2+x}{x^3-x}$ $f.$ $l(x)=\frac{1}{x^2+3x+2}$

g. $w(x)=\frac{2a^2+9x+1}{x^4+1}$

3.1.3 Basic operations of rational function

Addition, Subtraction, Multiplication and Division of rational expiration follow the same basic rules as addition, subtraction, multiplication and division of rational numbers. Those we find the definition of the above four fundamental operations on the rational functions analogous to those for rational numbers.

A. Addition and Subtraction of rational function

If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers, then $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$ $whereb\&d\neq 0.$

Similarly, If $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$ are two rational functions, then $\frac{p(x)}{q(x)}\pm\frac{r(x)}{s(x)}=\frac{p(x)s(x)\pm r(x)q(x)}{q(x)s(x}$,where $q(x)ands(x)\neq 0$

Example 1. Find the sum of $\frac{x-1}{x+2}$ and $\frac{x}{x-1}$

Solution:- $\frac{x-1}{x+2}+\frac{x}{x-1}$

Domain $=\{x: x+2andx-1\neq 0\}$

$=\{x: x+2\neq 0andx-1\neq 0\}$

$=\{x: x\neq 2andx\neq 1\}$

$=\{x: x\neq -2\&1\}$

$=\Re/\{-2,1\}$

Now, $\frac{x-1}{x+2}+\frac{x}{x-1}=\frac{(x-1)(x-1)+x(x+2)}{(x+2)(x-1)}=\frac{x^2-x-x+1+x^2+2x}{x^2-x+2x-2}=\frac{2x^2-2x+1+2x}{x^2+x-2}=\frac{2x^2+1}{x^2+x-2}$

Example2. Find the difference of $\frac{x^2+x}{x-1}$ and $\frac{x-3}{2x+1}$.

Solution:- $\frac{x^2+x}{x-1}-\frac{x-3}{2x+1}$

Domain $=\{x: x-1 \text{ and } 2x+1 \neq 0\}$

$=\{x: x-1 \neq 0 \text{ and } 2x+1 \neq 0\}$

$=\{x: x \neq 1 \text{ and } 2x \neq 1\}$

$=\left\{x: x \neq 1 \text{and } x \neq \frac{1}{2}\right\}$

$=\Re/\left\{\frac{1}{2},1\right\}$

Now, $\frac{x^2+x}{x-1}-\frac{x-3}{2x+1}=\frac{(x^2+x)(2x+1)-(x-3)(x-1)}{(x-1)(2x+1)}=\frac{2x^3+x^2+2x^2+x-(x^2-x-3x+3)}{2x^2+x-2x-1}$

$$=\frac{2x^3+3x^2+x-(x^2-4x+3)}{2x^2-x-1}=\frac{2x^3+3x^2+x-x^2+4x-3)}{2x^2-x-1}=\frac{2x^3+2x^2+5x-3}{2x^2-x-1}$$

Activity 3.8

Let $p(x)=\frac{1}{x+1}$ and $q(x)=\frac{x+1}{x-1}$ be two rational function, then

Find $p(x)+q(x)$ and $p(x)-q(x)$

B. Multiplication of Rational Function

Multiplication of rational functions also follows the same pattern as multiplication of rational Numbers.

If $\frac{a}{b}$ *and* $\frac{c}{d}$ are two rational numbers, then their product is defined as $\frac{a}{b} x \frac{c}{d}=\frac{ac}{bd}$ *where* $b \& d \neq 0$.

Similarly, multiplication of two rational functions, $\frac{p(x)}{q(x)}$ *and* $\frac{r(x)}{s(x)}$ are defined as:

$\frac{p(x)}{q(x)} x \frac{r(x)}{s(x)}=\frac{p(x)r(x)}{q(x)s(x}$, where $q(x)$ and $s(x) \neq 0$.

Example 1. Find the product of $\frac{x^2-1}{x}$ *and* $\frac{x+2}{x+3}$

Solution:-Domain= $\Re/\{-3,0\}$

Now,$\frac{x^2-1}{x}x\frac{x+2}{x+3}=\frac{(x^2-1)(x+2)}{x(x+3)}=\frac{x^3+2x^2-x-2}{x^2+3x}$

Exaple2. Find the product of$\frac{3x+4}{x+1}and\frac{x-2}{x+3}$

Solution:-Domain= $\Re/\{-1,-3\}$

Now,$\frac{3x+4}{x+1}x\frac{x-2}{x+3}=\frac{(3x+4)(x-2)}{(x+1)(x+3)}=\frac{3x^2-6x+4x-8}{x^2+3x+x+3}=\frac{3x^2-2x-8}{x^2+4x+3}$

C. Division of Rational Function

We know that the division of the rational number$\frac{a}{b}and\frac{c}{d}$does the follows:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}x\frac{d}{c}=\frac{ad}{bc}whereb,c\&d\neq 0.$$

Division of rational function is also defined in the same way of division of rational numbers.

The quotient of two rational function is the product of the first function and the reciprocal of the second function.

For two rational function,$\frac{p(x)}{q(x)}and\frac{r(x)}{s(x)}$ for,$\frac{r(x)}{s(x)}\neq 0$

$$\frac{p(x)}{q(x)}\div\frac{r(x)}{s(x)}=\frac{p(x)}{q(x)}x\frac{s(x)}{r(x)}=\frac{p(x)s(x)}{q(x)r(x)}$$

Example1: Divide A/ $\frac{x^2+2}{x}by\frac{x+2}{x}$

B/$\frac{x^2-3x-4}{x^2-1}by\frac{x^2-16}{x+3}$

Solution:-A.$\frac{x^2+2}{x}by\frac{x+2}{x}$

Domain= $\Re/\{0,-2\}$

$$\frac{x^2+2}{x}\div\frac{x+2}{x}=\frac{x^2+2}{x}x\frac{x}{x+2}=\frac{x^2+2}{x+2}$$

B/$\frac{x^2-3x-4}{x^2-1}by\frac{x^2-16}{x+3}$

Domain= $\Re/\{-4,-3,-1\&1\}$

$$\frac{x^2-3x-4}{x^2-1} \div \frac{x^2-16}{x+3} = \frac{(x+1)(x-4)}{(x-1)(x+1)} \div \frac{(x-4)(x+4)}{x+3}$$
$$= \frac{(x+1)(x-4)}{(x-1)(x+1)} \div \frac{x+3}{(x-4)(x+4)}$$
$$= \frac{x+3}{x^2+4x-x-4} = \frac{x+3}{x^2+3x-4}$$

Exercise 3.2

Perform the following operations and indicate the domain of the resulting expressions.

a. $\frac{x^2+1}{x-1} + \frac{x}{x-1} - \frac{x+1}{x-1}$ e.$\left(\frac{x^2-1}{x^2+3x-4}\right) + (\frac{x^2+x-12}{x^2+4x+3})$i. $\frac{x-2}{x+3} \div \frac{3x}{x^2+5x+6}$

b. $\frac{x+5}{2} + \frac{x-5}{2}$ f.$\frac{x^2-25}{(x-5)^2} \div \frac{x^2-16}{(x+4)^2}$

c. $\frac{x^2-3x-4}{x^2-1} \div \frac{x^2-16}{x+3}$g. $\frac{x^2+1}{x-1} + \frac{1}{x-1} - \frac{x+1}{x-1}$

d. $\frac{x}{x^2+1} + \frac{x^2+3x}{x^4-1} - \frac{1}{x-1}$ h.$\frac{x}{4-x} + \frac{xy+y^2}{x^2-y^2}$

3.1.4. Simplification of rational Expression

Definition :- A rational Expression is said to be simplified , when it is replaced by an equivalent rational expression which is in lowest terms,

i.e The numerator and the Denominator have only 1 as a greatest common factor.

Example 1:- Simplify$\frac{x^2-6x+8}{x^2-5x+6}$

Solution:-First write the denominators of the given polynomial as a product of simple polynomial factors.

i.e$x^2 - 6x + 8$ $and x^2 - 5x + 6$ can be written as $\frac{x^2-6x+8}{x^2-5x+6} = \frac{(x-2)(x-4)}{(x-2)(x-3)}$, canceling $x - 2$ in numerator and denominator, we get the given rational expression in its simplified form as$\frac{(x-4)}{(x-3)}$,$x \neq 2,3$

Example 2:- Simplify $\frac{\frac{x+1}{x-1} - \frac{x-1}{x+1}}{\frac{x}{x^2-1}}$

Solution: - The L.C.D of numerator is$(x-1)(x+1)$

This gives $\frac{\boldsymbol{x+1}}{\boldsymbol{x-1}} - \frac{\boldsymbol{x-1}}{\boldsymbol{x+1}} = \frac{\frac{(x-1)(x+1)-(x-1)(x-1)}{(x-1)(x+1)}}{\frac{x}{x^2-1}} = \frac{\frac{x^2+2x+1-(x^2-2x+1)}{x^2-1}}{\frac{x}{x^2-1}} = \frac{\frac{x^2+2x+1-x^2+2x-1)}{x^2-1}}{\frac{x}{x^2-1}}$

$$= \frac{\boldsymbol{4x}}{\boldsymbol{x}} = \boldsymbol{4}, \boldsymbol{x} \neq \boldsymbol{-1}, \boldsymbol{0}\&1$$

Exercise 3.3

Simplify the following rational Expression

a) $\frac{\frac{1}{x+1}-\frac{1}{x-1}}{\frac{1}{x+1}+\frac{1}{x-1}}$ b) $\frac{\frac{a^2-ab}{a+b}-\frac{1}{b-a}}{\frac{a+b}{b-a}-\frac{a}{a+b}}$ c) $\frac{\frac{x^2+3x-10}{5x-10}}{\frac{x^2-25}{x+5}}$

d) $\frac{\frac{1}{x-2}-\frac{2}{x+3}}{\frac{5}{x^2+x-6}-\frac{7}{x-2}}$ e) $\frac{x-1}{x-2} - \frac{x+1}{x+2} + \frac{x-6}{x^2-4}$ f) $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{x+1}}}}$

g) $1 + \frac{1}{1+\frac{1}{1+\frac{1}{x+\frac{1}{x+1}}}}$

3.1.5. Solving Rational Equations and Inequalities

A. Solving Rational Equation

Definition :-Rational equation is an equation which can be reducible in the form of $\frac{\boldsymbol{p(x)}}{\boldsymbol{q(x)}} = \boldsymbol{0}, where p(x) and q(x)$ are polynomial with $q(x) \neq 0$.

To solve any rational equation we have follow the following steps:

Step 1: Determine the universe (Domain) of the given rational Equation.

Step 2: Simplify the equation to an equivalent equation of the form $\frac{\boldsymbol{p(x)}}{\boldsymbol{q(x)}} = 0$, for some polynomials $p(x) and q(x)$.

Step 3: Solve the polynomial equation $p(x) = 0$.

Step 4: Find the intersection of the solution obtained in step3 with the universe of the equations.

Step5: The result obtained in step 4 is the solution set of the given rational equation.

Example 1: State the universe (Domain) and solve the following equations.

a) $\frac{x}{x-1} = \frac{1}{x-1}$

Solution: - The Domain $= \{x: x-1 \neq 0\}$

$$= \{x: x \neq 1\}$$

$$\therefore Domain = \Re/\{1\}$$

Now, $\frac{x}{x-1} = \frac{1}{x-1}$

$$\Rightarrow \frac{x}{x-1} - \left(\frac{1}{x-1}\right) = 0$$

$$\Rightarrow \frac{x}{x-1} - \frac{1}{x-1} = 0$$

$$\Rightarrow \frac{x-1}{x-1} = 0$$

$$\Rightarrow x - 1 = 0$$

$$\Rightarrow x = 1$$

But $\mathbf{1} \notin \Re/\{1\}$

Hence the solution set is $\emptyset$**.**

b) $\frac{-3x}{x+1} + \frac{6}{x} = \frac{3}{x+1}$

Solution:- The Domain $= \{x: x + 1 \neq 0, x \neq 0\&\}$

$$= \{x: x \neq -1, \&x \neq 0\}$$

$$\therefore Domain = \Re/\{-1,0\}$$

Now, $\frac{-3x}{x+1} + \frac{6}{x} = \frac{3}{x+1}$

$$\Rightarrow \frac{-3x(x) + 6(x+1)}{x(x+1)} = \frac{3}{x+1}$$

$$\Rightarrow \frac{-3x^2 + 6x + 6}{x(x-1)} = \frac{3}{x+1}$$

$$\Rightarrow \frac{-3x^2 + 6x + 6}{x(x-1)} - \frac{3}{x+1} = 0$$

$$\Rightarrow \frac{-3x^2 + 6x + 6 - 3(x)}{x(x-1)} = 0$$

$$\Rightarrow \frac{-3x^2 + 6x + 6 - 3x}{x(x-1)} = 0$$

$$\Rightarrow \frac{-3x^2 + 3x + 6}{x(x-1)} = 0$$

$$\Rightarrow -3x^2 + 3x + 6 = 0 \; both side divide - 3, then we get$$

$$\Rightarrow x^2 - x - 2 = 0$$

$\Rightarrow (x-2)(x+1) = 0$

$$\Rightarrow \boldsymbol{x} - \mathbf{2} = 0 \; and x + 1 = 0$$

$$\Rightarrow \boldsymbol{x} = \mathbf{2} and x = -1$$

But the solutions are 1 and 2

Hence the solution set $= Domain \cap solution = \Re/\{-1,0\} \cap \{\mathbf{1}, \mathbf{2}\} \therefore \boldsymbol{s}.\boldsymbol{s} = \{\mathbf{2}\}$

B. Solving rational Inequality

Definition:-An inequality that is equivalent to either $\frac{p(x)}{q(x)} \leq 0 \; or \frac{p(x)}{q(x)} < 0 \; or \frac{p(x)}{q(x)} \geq 0 \; or \frac{p(x)}{q(x)} > 0 \; or \frac{p(x)}{q(x)} \neq 0 \; for$ some polynomials$p(x) and q(x) and q(x) \neq 0 \; over real numbers$ is called Rational Inequality

N.B to solve rational inequality either case method or sign chart method.

I. Case Method

1. If the product of two quantities $p, q > 0, then$

Case I .$p > 0 \; and q > 0$

Case II.$p < 0 \; and q < 0$

1. If the product of two quantities$p, q \leq 0$,then

CaseI. $p \geq 0 \; and q \leq 0$

Case II.$p \leq 0 \; and q \geq 0$

2. If the quotient of two quantities$\frac{p}{q} > 0, then$

CaseI. $p \geq 0 \; and q \geq 0$

Case II.$p \leq 0 \; and q \leq 0$

4. If the quotient of two quantities $\frac{p}{q} \leq 0, then$

CaseI. $p \geq 0 \; and q \leq 0$

Case II.$p \leq 0 \; and q \geq 0$

Example1:- Solve the following rational inequality by using case method.

a) $\frac{x}{x-3} > 0$

Solution: to solve the rational inequalities

Let us consider two cases

Domain= $\Re/\{1\}$

Case 1:$x > 0\ and x - 3 > 0, since \frac{x}{x-3} > 0$

$\Rightarrow \boldsymbol{x} > 0\ and x > 3$

$\Rightarrow (\mathbf{0}, \infty) \cap (\mathbf{3}, \infty) = (\mathbf{3}, \infty)$

$\boldsymbol{Thus}, \boldsymbol{s}.\boldsymbol{s_1} = \Re/\{3\} \cap \{\mathbf{3}, \infty\} = \{\mathbf{3}, \infty\}$
$= \{\boldsymbol{x}: \boldsymbol{x} > 3\}$

Case2:$x < 0\ and x - 3 < 0$

$\Rightarrow x < 0\ and x < 3$

$\Rightarrow (-\infty, \mathbf{0}) \cap (-\infty, \mathbf{1}) = (-\infty, \mathbf{0})$

Thus $\boldsymbol{s}.\boldsymbol{s_2} = \Re/\{3\} \cap (-\infty, \mathbf{0}) = (-\infty, \mathbf{0}) = (-\infty, \mathbf{0})$

There fore , $\boldsymbol{s}.\boldsymbol{s} = \boldsymbol{s}.\boldsymbol{s_1} \cup \boldsymbol{s}.\boldsymbol{s_2} = (-\infty, \mathbf{0}) \cup (\mathbf{3}, \infty)\boldsymbol{or}$

$\boldsymbol{s}.\boldsymbol{s} = \{\boldsymbol{x}: \boldsymbol{x} < 0\ or x > 3\}$

b) $\frac{\boldsymbol{x}}{\boldsymbol{x+2}} \leq \mathbf{0}$

Solution:-To solve the rational inequalities

Let us consider two cases

Domain= $\Re/\{2\}$

Case 1:$x \geq 0\ and x + 2 \leq 0$

$\Rightarrow x \geq 0\ and x \leq -2$

$\Rightarrow [0\ , \infty) \cap (-\infty, -2]$

$\Rightarrow s.s = \emptyset$

Thus $\boldsymbol{s}.\boldsymbol{s_1} = \Re/\{2\} \cap \emptyset = \emptyset$

Case 2:$x \leq 0\ and x + 2 \geq 0$

$\Rightarrow x \leq 0\ and x \geq -2$

$\Rightarrow (-\infty, 0] \cap [-2, \infty) = [-2,0]$

Thus $\boldsymbol{s.s_2} = \frac{\Re}{\{2\}} \cap = [-2,0] = [-2,0]$

Therefore, $s.s = \boldsymbol{s.s_1} \cup \boldsymbol{s.s_2} = \emptyset \cup [-2,0] = [-2,0]$

Activity 3.9

Q1: state the universe and solve the following rational inequality.

A) $\frac{x+1}{x-1} > 0$ b) $\frac{x}{x-2} \leq 0$ c) $1 \leq \frac{x}{x+4} \leq 3$ D) $\frac{x^2-x+6}{x^2+5x+6} \geq 0$

II**. Sign Chart Method**

Inequities can also be solved by a more efficient (best) method known as **sign chart method**.

When we solve rational inequalities using sign chart, we should consider the following steps:-

Step1: simplify the inequalities such that the right side of the inequality is zero.

Step 2: Factorize both the numerator and the denominator and state the universe.

Step3: Find the zeros of the numerator and the denominator.

Step4: write the zeros in ascending order on a horizontal line to determine the interval.

Step 5: Find the algebraic sign of each factor between the intervals.

Step6: Find the algebraic sign of the ratio or the product of factor between the intervals.

Step7: Find the solution set of the inequality according to the question at hand and algebraic sign obtained in step 6 by considering the domain.

Example 1: solve the following rational inequality.

A) $\frac{(x-3)(x+3)}{(x+1)(x-2)} > 0$ B)$\frac{x^4-16}{-x^2+x+6} \geq 0$

Solution: a) the domain of the inequality $\frac{(x-3)(x+3)}{(x+1)(x-2)} > 0$ is areal number except$-1\ and\ 2$.

The zeros of the numerator are t$-3\ and\ 3$ and the zeros of the denominator are $-1\ and\ 2$.

Now, we can construct a sign chart as

	-3		-1		2		3	
$x-3$- -	-	- -	-	------	-	- -	0	+ +
$x+3$- -	0	+	+	+ +	+	+ +	+	+ +
$(x-3)(x+3)$ +	0	- -	-	- -	-	- -	0	
$x+\mathbf{1}$ - -	-	- -	0	+ +	+	+ +	+	+ +
$x-2$- -	-	- -	-	- -	0	+ +	+	+ +
$x+1)(x-2)$ + +	+	+	0	- -	0	+ +	+	+ +
$\frac{(x-3)(x+3)}{x+1)(x-2)}$ + +	0		∄	+ +	∄	+ +	0	+ +

The expression $\frac{(x-3)(x+3)}{(x+1)(x-2)}$ is positive on the interval $(-\infty,-3)\cup(-1,2)\cup(3,\infty)$.

Therefore, the solution set of the inequality is

$$\therefore s.s=(-\infty,-3)\cup(-1,2)\cup(3,\infty)$$

B) The numerator x^2-16 is factorized in to $(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$and the denominator $-x^2+x+6$ is factorized in to $(x+2)(3-x)$.

The domain of inequality $\Re/\{-2,3\}$

Thus,$\frac{x^4-16}{-x^2+x+6}=\frac{(x-2)(x+2)(x^2+4)}{(x+2)(3-x)}\geq 0$The zeros of the numerator are-2 *and* 2 *andzero's*

$$denomnatorare-2\ and3$$

Now, we can construct a sign chart as shown below;

	- 2		2		3	
x^2+4 + +	+	+ +	+	+ +	+	+ +
$x-2$ - -	-	- -	0	+ +	+	+ +
$x+2$ - -	0	+ +	+	+ +	+	+ +
$(x^2+4)(x-2)(x+2)$+ +	0	- -	0	+ +	+	+ +
$x+2$ - -	0	+ +	+	+ +	+	+ +
$(x-3)$ + +	+	+	+	+ +	0	- -
$(x+2)(x-3)$ - -	0	+ +		+ +	0	- -
$\frac{(x^2+4)(x-2)(x+2)}{(x+2)(x-3)}$ - -	∄	+ +	0	+ +	∄	- -

The expression $\frac{(x-2)(x+2)(x^2+4)}{(x+2)(3-x)}$ is greater than or equal to zero on the interval $[2,3)$.

Therefore, the solution set of the inequality is

$$\therefore s.s = [2,3] or s.s = \{x: 2 \leq x < 3\}$$

Exercise3.10

Q1. Solve the following inequalities by using sign chart.

a) $\frac{8}{x^2-9} + \frac{x-1}{x^2+8x} > \frac{1}{x-4}$ b) $\frac{2}{x^2-1} - \frac{2}{x^2-x-2} \leq \frac{x}{x^2-3x+2}$ c) $-1 + \frac{1-x}{3} \leq \frac{2}{5}$ d) $\frac{2x^2-x-1}{x^2-4x+4} \leq 0$

3.1.6. Sketch the graphs of a given rational function

To draw the graph of rational functions, we should give great emphasis for the following terms. There are intercept, symmetry and an asymptote.

I. Intercept

There are two kind of intercept s namely, x- intercept and y-intercept.

The x- intercept is the point at which the graph crosses the x-axis and y- intercept is the coordinate the point at which the graph crosses the y-axis,

II Asymptote

Definition:- An asymptote is the line to which the graph of the function approaches for extreme values of the domain.

Asymptote is not the part of the graph.

There are three kind of the asymptote such as

- **I) Vertical asymptote**
- **ii) Horizontal asymptote and**
- **iii) Oblique asymptote**

I).Vertical asymptote

Definition1:- For any rational function $R(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$.

If $q(a) = 0$ and $(x - a)$ is not a factor of p(x) then the vertical line $x = a$ is called the vertical asymptote(s).

Note: i) If $q(a) \neq 0$ for all a in the domain of $R(x)$, *then* the graph does not have a vertical asymptote.

ii) The graph of a rational function can have more than one vertical asymptote.

Example1: Find the vertical asymptote of the following rational functions.(without sketching the graph).

$a)\ .f(x) = \frac{x}{x^2-4}$b) . $g(x) = \frac{x}{x^3-2x}$ c). h $(x) = \frac{x-1}{x^2+1}$

Solution: - a) The denominator of f(x) is$x^2 - 4$. Then

$$\Rightarrow x^2 - 4 = 0$$

$$\Rightarrow (x - 2)(x + 2) = 0$$

$$\Rightarrow x - 2 = 0 \text{ or } x + 2 = 0$$

$\Rightarrow x = 2$ or $x = -2$

Hence, the domain of $f(x)$ is $\Re/\{-2,2\}$.

Therefore, $\boldsymbol{x} = 2$ and $x = -2$ are vertical asymptotes of f(x).

b). The denominator of g(x) is$x^3 - 2x$. Then

$$\Rightarrow x^3 - 2x = 0$$

$$\Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0 \text{ or } x^2 - 2 = 0$$

$\Rightarrow x = 0$ or $x = \pm\sqrt{2}$. But $x = 0$ is a factor of the numerator and the denominator

Hence, the domain of $f(x)$ is $\Re/\{0, \pm\sqrt{2},\}$.

Therefore, $\boldsymbol{x} = \sqrt{2}$ and $x = -\sqrt{2}$ are the only vertical asymptotes of g(x).

C). The denominator of h(x) is$x^2 + 1$ .Then

$$\Rightarrow x^2 + 1 = 0$$

$\Rightarrow x^2 = -1$, but the square of any real number x is non negative .

Hence, the domain of $h(x)$ is the set of all real numbers.

Therefore;$thegraphofh(x)has\ no$vertical asymptote.

N.B. 1. If $D(x) \neq 0$ for all $x \in \Re$ in the domain of F(x), then the graph does not have a vertical asymptote.
2. The graph of f(x) never intersects its vertical asymptote.

II. Horizontal Asymptote

Definition:- For any rational function $R(x) = \frac{\mathbf{p(x)}}{\mathbf{q(x)}}, q(x) \neq 0$.

A. If the degree of numerator **,** $\mathbf{p(x)}$ is less than the degree of the denominator $q(x)$, then the horizontal line$y = 0$(x-axis) is called horizontal asymptote of the graph of $R(x)$.

B. If the degree of numerator **,** $\mathbf{p(x)}$ is equal to the degree of the denominator $\mathbf{q(x)}$, then the horizontal line$y = \frac{a}{b} iscalledahorizontalasymtoteofthegraphofR(x)$;

Where **a** is leading coefficient of$\mathbf{p(x)}$and b is leading coefficient of$\mathbf{q(x)}$.

Example1: Find the Horizontal asymptote of the following rational functions. (without sketching the graph).

A) $\mathbf{f(x)} = \frac{\mathbf{x^2+1}}{\mathbf{x^3-x}}$ **B)** $\boldsymbol{g}\mathbf{(x)} = \frac{\mathbf{2x^2+1}}{\mathbf{4x^2-4}}$

Solution : A) the degree of numerator $x^2 + 1$, is 2 and the degree of denominator

$$\mathbf{x^3 - x \text{ is } 3}$$

This shows that the degree of Numerator is greater than the degree of denominator,
Therefore, the horizontal line $y = 0$ is the horizontal asymptote**.**

B) The degree of numerator $2x^2 + 1$, is 2 and the degree of denominator

$4x^2 - 4$ is 2.

This shows that the degree of Numerator is equal to the degree of denominator,
The horizontal asymptote obtained by dividing the leading coefficient of numerator by the leading coefficient of denominator

Therefore, the horizontal asymptote **is**$\boldsymbol{y} = \frac{\mathbf{1}}{\mathbf{2}}$ **.**

III. Oblique Asymptotes

Definition: for any rational function$R(x) = \frac{\mathbf{p(x)}}{\mathbf{q(x)}}, q(x) \neq 0$.
If the degree of numerator**,** $\mathbf{p(x)}$ is one higher than the degree of the denominator $\mathbf{q(x)}$, then the long division we obtain $R(x) = ax + b + \frac{r(x)}{q(x)}$,
where $a \neq 0$ $anda, b, \in \Re$
and the degree of r(x) is less than the degree of q(x) and the oblique line y(x)=ax+b is called oblique asymptote of the graph of R(x).

Note that:-i. A straight line is said to be oblique if it is neither vertical nor horizontal line.

ii. The graph of f(x) may cross its horizontal or oblique asymptote.

Example1: Find the oblique asymptote of the following rational functions.(without sketching the

graph). A) . $f(x) = \frac{2x^2-1}{x+1}$

Solution: $f(x) = \frac{2x^2-1}{x+1}$ can be written as $f(x) = \frac{2x^2-1}{x+1} = (2x-2) + \frac{1}{x+1}$ by using long division.

Thus the line $y = 2x - 2$ is oblique asymptote of the graph of f(x).

To sketching the graph of rational functions we can follow the following procedure.

i. Factorize the numerator $p(x)$ and the denominator $q(x)$

And reduce the fraction $\frac{p(x)}{q(x)}$ by canceling all common factors.

ii. Find the intercepts of the graph of $R(x)$, if their exist.

iii. Find the asymptote of the graph of $R(x)$, if their exist.

iv. use the x-intercepts and vertical asymptotes to divide the x-axis interval, Determine the algebraic sign of $R(x)$ on each of this interval; this show where the graph of R(x)lies above or below the x-axis

v. Determine the symmetry of the graph of the function.

vi. Sketch the graph of $R(x)$ by using the information obtained in the above all steps.

Example: Sketch the graph of each of the following rational function.

a/ $R(x) = \frac{x}{x^2-4}$ b/ $R(x) = \frac{x^2-1}{x^2-4}$ c. $R(x) = \frac{x^2-2}{x-2}$

Solution: a). Domain of $R(x) = \Re/\{-2,2\}$

i. Factorize the numerator and the denominator gives:

$$R(x) = \frac{x}{x^2-4} = \frac{x}{(x-2)(x+2)}$$

ii. Find intercepts (i. e x and y intercept).

- To find x- intercept put $y = R(x) = 0$. then $R(x) = 0$

$\Rightarrow \frac{x}{(x-2)(x+2)} = 0 \quad \Rightarrow x = 0$

Hence, the graph intercepts then $x -$ axis at a point (0,0).

- To find y- intercept put $x = 0$. then $f(0) = 0$
  $\Rightarrow$ence , the graph intercepts then $y -$ axis at a point (0,0).

iii. Find the asymptotes.

- The denominator of f(x) is zero at the point $x = -2\ or x = 2$.
  Hence, $\boldsymbol{x = -2 and x = 2 are}$**thevertical asymptotes** of the graph of R(x)
- The degree of the numerator is less than the degree of denominator.
  Hence, $\boldsymbol{y = 0}$ **is Horizontal asymptote of** the graph of R(x).

iv. To show where the graph of R(x)lies above or below the x-axis. Determine the algebraic sign of R(x) on each of this interval we use sign chart as follows.

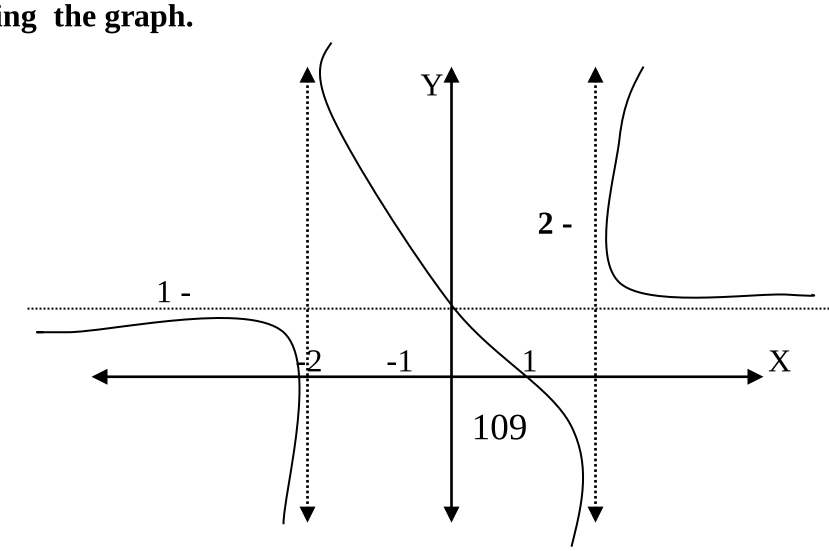
	-2		0		2	
$x-2$- -	0	+	+	+ +	+	+ +
$x+2$- -	-	-	-	- -	0	+ +
x - -	-	- -	0	+ +	+	+ +
$(x-2)(x-2)$+ +	0	- -	-	- -	0	+ +
$\frac{x}{(x-2)(x+2)}$ –	∄	+	0	- -	∄	+ +

For $x \in (-2,0] \cup (2,\infty), R(x) > 0$.

That means the graph of $R(x)$ lies above the$x - axis$ on this interval.

- For $x \in (-\infty - 2) \cup [0,2), R(x) < 0$.
  That means the graph of $R(x)$ lies below the$x - axis$ on this interval.
- Since $R(x)$ is an odd function,
  i.e $R(-x) = -R(x)$, the graph of $R(x)$ is symmetric about the origin.

v. **sketching the graph.**



-1-

-2 –

b).Domain of $R(x) = \Re/\{-2,2\}$

i. Factorize the numerator and the denominator gives:

$$R(x) = \frac{x^2 - 1}{x^2 - 4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}$$

ii. Find intercepts (i. e x and y intercept).

- To find x- intercept put $y = R(x) = 0$. then $R(x) = 0$

  $\Rightarrow \frac{(x-1)(x+1)}{(x-2)(x+2)} = 0$

$$\Rightarrow x^2 - 1 = 0$$
$$\Rightarrow x = \pm 1$$

Hence , the graph intercepts then x − axis at a point (-1,0) and (1,0).

- To find y- intercept put $x = 0$. then $R(0) = \frac{1}{4}$

  Hence , the graph intercepts then y − axis at a point $(0, \frac{1}{4})$.

iii. Find the asymptotes.

- The denominator of f(x) is zero at the point $x = -2$ or $x = 2$.
  Hence, $x = -2$ or $x = 2$ are the **vertical asymptotes** of the graph of R(x)
- The degree of the numerator is equal to the degree of denominator.
  Hence, dividing the leading coefficient of the numerator by the leading coefficient of the denominator gives $R(x) = 1 + \frac{3}{x^2-4}$.
  Thus, $R(x) = 1$ is a**Horizontal asymptote** of the graph of R(x).

iv. To show where the graph of R(x)lies above or below the x-axis.determine the algebraic sign of R(x) on each of this interval we use sign chart as follows.

	-2		-1		1		2	
$x-2$ - -	-	-	-	- -	-	- -	**0**	+
$x+2$ - -	0	+	+	+ +	+	+ +	+	+
$(x-2)(x+2)$ + +	0	- -	-	- -	-	- -	0	+
$\boldsymbol{x-1}$ **- -**	-	- -	-	- -	0	+ +	+	+
$x+1$ - -	-	- -	0	+ +	+	+ +	+	+
$(x-1)(x+1)$ + +	+	+ +	0	-	0	+ +	+	+
$\frac{(x-2)(x+2)}{(x-1)(x+1)}$ + +	∄	-	0	+ +	0	- -	∄	+

- For $x \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty) R(x) > 0$.

That means the graph of $R(x)$ lies above the $x - axis$ on this interval.

- For $x \in (-2, -1] \cup [1,2), R(x) < 0$.

  That means the graph of $R(x)$ lies below the $x - axis$ on this interval.

- Since $R(x)$ is an even function,

  i.e $R(x) = R(-x)$, the graph of $R(x)$ is symmetric about the y- axis.

v. sketch the graph

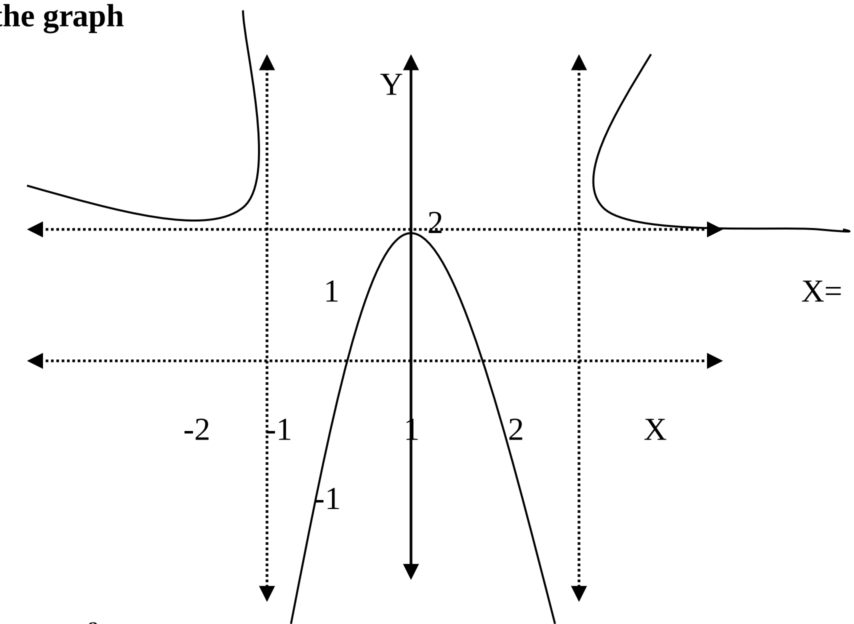


Fig. Graph of $f(x) = \frac{x^2-1}{x^2-4}$

C) Domain of $R(x) = \Re/\{2\}$

i. Factorize the numerator and the denominator gives:

$$R(x) = \frac{x^2 - 2}{x - 2} = \frac{(x - \sqrt{2})(x + \sqrt{2})}{(x - 2)}$$

ii. Find intercepts (i. e x and y intercept).

- To find x- intercept put $y = R(x) = 0$. then $R(x) = 0$

  $\Rightarrow \frac{(x-\sqrt{2})(x+\sqrt{2})}{x-2} = 0$

  $\Rightarrow (x - \sqrt{2})(x + \sqrt{2}) = 0$

  $\Rightarrow x - \sqrt{2} = 0$ and $x + \sqrt{2} = 0$

  $\Rightarrow x = -\sqrt{2}$ and $x = \sqrt{2}$

Hence, the graph intercepts then x – axis at a point $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.

- To find y- intercept put $x = 0$. then $R(0) = 1$

Hence, the graph intercepts then $y-$ axis at a point $(0,1)$.

iii. Find the asymptotes.

- ❖ The denominator of f(x) is zero at the point $x=2$.
  Hence, $x=2$ *is* the **vertical asymptote** of the graph of R(x).
- ❖ The degree of the numerator is one higher than the degree of denominator, by long division $R(x)=x+2+\frac{2}{x-2}$.
  Hence, it has no **Horizontal asymptotes** of the graph of R(x) but an oblique asymptote $y=x+2$.

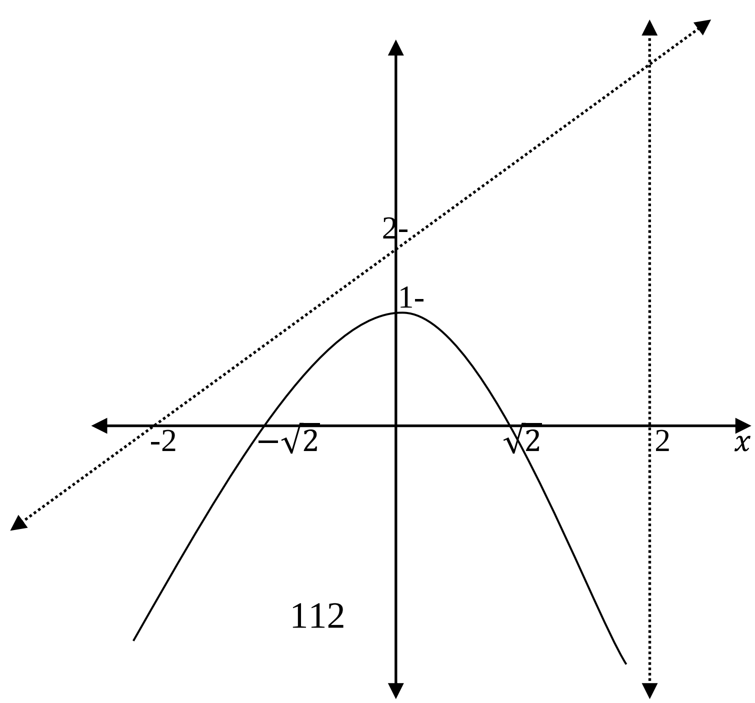
iv. To show where the graph of R(x)lies above or below the x-axis.determine the algebraic sign of R(x) on each of this interval we use sign chart as follows.

	$-\sqrt{2}$		$\sqrt{2}$		2	
$x-\sqrt{2}$- -	0	+ +	+	+ +	+	+ +
$x+\sqrt{2}$ - -	-	-	0	+ +	+	+ +
$(x-\sqrt{2})(x+\sqrt{2})$ +	0	- -	0	+ +	+	+ +
$\boldsymbol{x-2}$ - -	-	- -	-	- -	0	+ +
$\frac{(x-\sqrt{2})(x+\sqrt{2})}{x-2}$ - -	0	++	0	- -	∄	+ +

For $x\in[-\sqrt{2},\sqrt{2}]\cup(2,\infty)R(x)>0$.

That means the graph of $R(x)$ lies above the$x-axis$ on this interval.

- ➢ For $x\in(-\infty,\sqrt{2}]\cup[\sqrt{2},2),R(x)<0$.
  That means the graph of $R(x)$ lies below the$x-axis$ on this interval.
- ➢ Since $R(x)$ is an even function,$R(x)=R(-x)$, the graph of $R(x)$ is Symmetric about the y- axis.



Exercise3.11

1. Sketch the graph of the following.

a) $f(x) = \frac{x^3-2x-x+2}{x^2-1}$ b) $f(x) = \frac{x^3+x^2-6x}{x^2-x-2}$ c). $f(x) = \frac{3x^2+x+1}{x^2-1}$

$d).\, f(x) = \frac{x+1}{x^2+x-2}$ $e).\, f(x) = \frac{x^2-4}{x^2-1}$ f). $f(x) = \frac{x^2-x-6}{x-2}$

g). $f(x) = \frac{x^2+2x+1}{x+1}$

3.2 Exponential Function

Introduction

The mathematics of logarithms and exponentials occurs naturally in many branches of science. It is very important in solving problems related to growth and decay. The growth and decay may be that of a plant or a population, a crystalline structure or money in the bank. Therefore we need to have some understanding of the way in which logarithms and exponentials work.

This unit defines and investigates exponential and logarithmic functions. We motivate exponential functions by their wide variety of application. We introduce logarithmic functions as the inverse functions of exponential functions and exploit our previous knowledge of inverse functions to investigate these functions. In particular, we use this inverse relationship for the purpose of solving exponential and logarithmic equations

Revision on Positive Integral Exponent

For a natural number n and a positive integer a; then the expression "a^n" read as "the n^{th} power of a" or "a raised to n" and defined as:

$a^n = \underbrace{a.a.a.\dots.a}$ (The product of n equal factors; each factor equal to a)

In the symbol a^n, a is called the base and n is called the exponent

Example; $3^5 = 3.3.3.3.3 = 243$

$$(\sqrt{2})^4 = \sqrt{2}.\sqrt{2}.\sqrt{2}.\sqrt{2} = \sqrt{2.2.2.2} = \sqrt{16} = 4$$

$$(\frac{2}{3})^3 = \frac{2}{3}.\frac{2}{3}.\frac{2}{3} = \frac{2.2.2}{3.3.3} = \frac{8}{27}$$

3.2.1 Revision on Rules of Exponents

Let a and b are any real numbers, m and n are positive integers;

1. **Product Rule with the Same Base but Different Exponent**

To multiply powers with the same base, keep the base and add the exponents

$$a^m x a^n = a^{m+n}$$

Proof:$-a^m x a^n = \underbrace{a.a.a.\dots.a}_{m\ factors}\ x\ \underbrace{a.a.a.\dots.a}_{n\ factors} = \underbrace{a.a.a.\dots.a}_{m+n\ factors} = a^{m+n}$

2. **Product Rule with the Same Exponent but Different Base**

Suppose now the bases are different but the exponents are the same; using the commutative and associative properties of multiplication in the system of real numbers.

$$(ab)^n = a^n x b^n$$

Proof:- $(ab)^n = \underbrace{ab\ x\ ab\ x\ ab\ x \dots xab}_{n\ factors} = \underbrace{a\ x\ a\ x\ a\ x \dots xa}_{n\ factors}\ x\ \underbrace{b\ x\ b\ x\ b\ x \dots xb}_{m\ factors} = a^n x b^n$

3. **power Rule**

To simplify power of power, keep the base and multiply the exponents

$$(a^n)^m = a^{mn}$$

Proof;$-(a^n)^m = \underbrace{a^n x a^n x a^n x \dots x a^n}_{m\ factors} = a^{\overbrace{n+n+n+\dots+n}^{m\ factors}} = a^{nm}$

4. **The Quotient Rule with the same base but different exponent.**

To divide powers with the same base, keep the base and subtract the exponents

$$\frac{a^n}{a^m} = a^{n-m}, \qquad a \neq 0$$

Proof:- $If n > m\ and\ a \neq 0, then$

$$\frac{a^n}{a^m} = \frac{a^{n-m} x a^m}{a^m} = a^{n-m}$$

$If\ m = n, then\ 1 = \frac{a^n}{a^n} = a^{n-n} = a^0$

$a^0 = 1$, for any $a \neq 0$

$If\ n < m, \qquad \frac{a^n}{a^m} = a^{n-m} = a^{-k} = \frac{1}{a^{m-n}} = \frac{1}{a^k}, for\ k = m - n > 0$

$i.e$ For any nonzero real number ‘a’ and natural number ‘n’,

$$a^{-n} = \frac{1}{a^n} \Leftrightarrow a^n = \frac{1}{a^{-n}} and\ a^{-n}\ is\ the\ multiplicative\ inverse\ of\ a^n$$

5. **The Quotient Rule with the Same Exponent but Different Base**

To divide powers with the same exponent but different base, keep the exponent and divide the base.

$$\frac{a^n}{b^n} = (\tfrac{a}{b})^n,\ b \neq 0 \Leftrightarrow (\tfrac{a}{b})^n = \frac{a^n}{b^n} \quad \text{and} \quad (\tfrac{a}{b})^{-n} = (\tfrac{b}{a})^n,\ \ a \neq 0, b \neq 0$$

Proof:- $\frac{a^n}{b^n} = \frac{\overbrace{axaxax\ldots xa}^{\text{n factors}}}{\underbrace{bxbxbx\ldots xb}_{\text{n factors}}} = \underbrace{\frac{a}{b}x\frac{a}{b}x\frac{a}{b}x\ldots x\frac{a}{b}}_{\text{n factors}} = (\frac{a}{b})^n, b \neq 0$ or

$$(\frac{a}{b})^n = (a.\frac{1}{b})^n = a^n.(\frac{1}{b})^n = a^n.\frac{1}{b^n} = \frac{a^n}{b^n}, b \neq 0$$

Note:- The above five rules are called the rules of exponents for positive integral exponents

The rules of exponents help us in simplifying expressions that involve powers.

Example1: simplify the following

a. $(x^4y^{-5})^2 = (x^4)^2(y^{-5})^2 = x^{4x2}y^{-5x2} = x^8y^{-10} = \frac{x^8}{y^{10}}$

b. $\frac{9c^{-5}d^5}{18c^{-7}d^{-4}} = \frac{9c^{-5-(-7)}d^{5-(-4)}}{9*2} = \frac{c^2d^9}{2}$

c. $\left(\frac{27y^3z^{-6}}{x^{-3}}\right)^{\frac{-1}{3}} = \left(\frac{3^3y^3z^{-6}}{x^{-3}}\right)^{\frac{-1}{3}} = \frac{(3^3y^3z^{-6})^{\frac{-1}{3}}}{(x^{-3})^{\frac{-1}{3}}} = \frac{(3^3)^{\frac{-1}{3}}(y^3)^{\frac{-1}{3}}(z^{-6})^{\frac{-1}{3}}}{x^{-5*\frac{-1}{3}}} = \frac{\left(3^{3\,x\frac{-1}{3}}\right)\left(y^{3\,x\frac{-1}{3}}\right)\left(z^{-6\,x\frac{-1}{3}}\right)}{x^{\frac{5}{3}}}$

$$= \frac{3^{-1} * y^{-1} * z^2}{x^{\frac{5}{3}}} = \frac{z^2}{3yx^{\frac{5}{3}}}$$

Activity3.10

Simplify the following expressions

a. $(\frac{xy^3z^{-4}}{x^{-3}y^{-2}z^2})^{-2}$ b. $\frac{x^nx^{3n+5}}{x^{4n}}$

Zero and Negative Integral Exponents

Let us discuss about the meaning of power of non-zero base, with

i. **Zero Exponent**

For all positive integer 'n' and any real number 'a'

$$a^0.a^n = a^{0+n} = a^n$$

$$a^n.a^0 = a^{n+0} = a^n$$

but we know that 1 *is the unique multiplicative identity*

$i.e\ 1.a^n = a^n = a^n.1$

Accordingly we get the following definition

$If\ a \in \mathbb{R}\ and\ a\ \neq 0\ then$

$$a^0 = 1$$

ii. **Negative Integral Exponent**

If 'n' is a positive integer, then '–n' is a negative integer. Product rule with the same Base but different exponent;

$$a^n.a^{-n} = a^{n+(-n)} = a^0 = 1$$

This implies a^{-n} is the multiplicative inverse of a^n. But $\frac{1}{a^n}$ is also a multiplicative inverse of a^n, then by uniqueness of multiplicative inverse a^{-n} *must be equal to* $\frac{1}{a^n}$

$\boldsymbol{i.e}$ *If* $a \in \mathbb{R}$, $a \neq 0$ and n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

Note; If a = 0 and n is positive integer, then

- $0^{-n} = \frac{1}{0^n} = \frac{1}{0}$ which is undefined
- $0^{-n} = 0^{1+(-1)} = 0^1.0^{-1} = 0.\frac{1}{0^1} = \frac{0}{0}$, Still this is undefined.

Definition:- If a and b are real numbers and n is a positive integer greater than 1 such that $a^n = b$ then a is called the n^{th} root of b

Example1:- 2 is the sixth root of 64 b/c 2^6=64.

-2 is also the sixth root of 64 b/c $(-2)^6 = 64$.

-3 is the fifth root of -243 b/c $(-3)^5 = -243$.

Definition:- If a is real number and n is a positive integer greater than 1, then the principal n^{th} rootof a is denoted by $\sqrt[n]{a}$ and defined as

- The positive n^{th} root of a if a>0 and n is even
- The negative n^{th} root of a, if a<0 and n is odd.
- 0, if a=0

Example1

- ✓ 6 is the principal fourth root of 1296, -6 is the fourth root of 1296 but not the principal fourth root of 1296.
- ✓ -3 is the principal seventh root of -2187.

Rational Exponents

If n and m are integers, we now that$(a^n)^m = a^{mn}$. This rule is still holds when $n = \frac{1}{m}$

$$i.e\ (a^{\frac{1}{m}})^m = a^{mx\frac{1}{m}} = a^1 = a$$

Definition: 1. $If\ a \in \mathbb{R}\ and\ n\ with\ n \neq 1\ is\ a\ postive\ integer, then$

$$(a)^{\frac{1}{n}} = \sqrt[n]{a}, \qquad when\ \sqrt[n]{a}\ is\ a\ real\ number$$

2.$If\ a \in \mathbb{R},\ n\ and\ m\ are\ \ postive\ integers, then$

$$(a)^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Provided $\sqrt[n]{a}$ is a real number

Note: the exponential rules we discussed for integral exponents also hold for rational exponents and also hold for real exponents.

Rules of Radical

If a and b are real numbers and n and m are positive integer greater than 1, then

- $\sqrt[n]{a}\,.\sqrt[n]{b} = \sqrt[n]{ab}$, if n is even then a and b are non-negative real numbers
- $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},\ b \neq o$,if n is even then a and b are non-negative real numbers.
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$,if nm is even then a is non-negative real number.

Example; simplify;

a. $\sqrt[3]{81}\,.\sqrt[3]{9} = \sqrt[3]{(81)(9)} = \sqrt[3]{729} = 9$

b. $\frac{\sqrt[4]{512}}{\sqrt[4]{8}} = \sqrt[4]{\frac{2048}{8}} = \sqrt[4]{256} = 4$

c. $\sqrt[9]{19{,}683} = \sqrt[3x3]{19{,}683} = \sqrt[3]{\sqrt[3]{19{,}683}} = \sqrt[3]{27} = 3$

Activity3.11

1. Simplify the following expressions

a. $(\frac{x^{-2}y^{\frac{-1}{6}}}{x^{-4}y})^3$ b. $\frac{15x^{-3}y^2z^{-4}}{20x^{-4}y^{-3}z^2}$

2. Simplify each of the following

a. $\frac{\sqrt{40}}{\sqrt{10}}$ b. $\sqrt{3xy}.\sqrt{12xy^3}\ for\ \ x, y \geq 0$

3.2.2. Definition of Exponential Function

Functions given by the expressions of the form

$$\boldsymbol{y = f(x) = a^x,}$$

Where 'a' is a fixed positive number and$a \neq 1$, is called **exponential function**. The number '*a*' is called the **base** of the exponential function.

Every exponential function of this form has all real numbers as its domain and all positive realnumbers as its range. Exponential functions can be evaluated for integer values of x by inspection orby arithmetic calculation, but for most values of x, they are best evaluated with a scientific calculator.

The following are some examples of exponential functions

$$f(x) = 2^x, g(x) = \left(\frac{2}{3}\right)^x, h(x) = 10^x, \; etc$$

The following are properties of an exponential function and are useful in drawing their graphs.

- If a = 1, then $f(x) = a^x = 1^x = 1$ for all real number x, and hence the graph of $y = 1^x$ is the line y = 1
- Since $a^0 = 1$, for $a > 0$, it follows that the graph of any exponential function passes through the point (0, 1).
- If $a > 1$ then $a^x > 1$ for all $x > 0$ and in facta^x keeps increasing without bound as x increases. on the other hand for $x < 0$, we get $0 < a^x < 1$ and the graph approaches to the negative x-axis.
- If $0 < a < 1$, then $0 < a^x < 1$ for all x > 0, and $for\ x < 0\ we\ have\ a^x > 1$ in fact a^x keeps increasing without bound as x goes to negative infinity and the graph approaches to the positive x-axis
- The graph of g(x) = $(\frac{1}{a})^x$is the reflection of the graph of y = a^xalong the y-axis.

Naturally, a larger value of ‘a’ will cause the graph y = a^x to rise more rapidly.

3.2.2.1 The graph of Exponential Function

Steps in graphing Exponential Function

1. Establish a table of values by considering the function in the form of $y = a^x$
2. Plot points from the table of values on the coordinate axis
3. Connect points with a smooth curve to form the graph.

Example1; draw the graph of

$f(x) = 2^x$

First choose convenient value for x, let $x = -3, -2, -1, 0, 1, 2\ and\ 3$

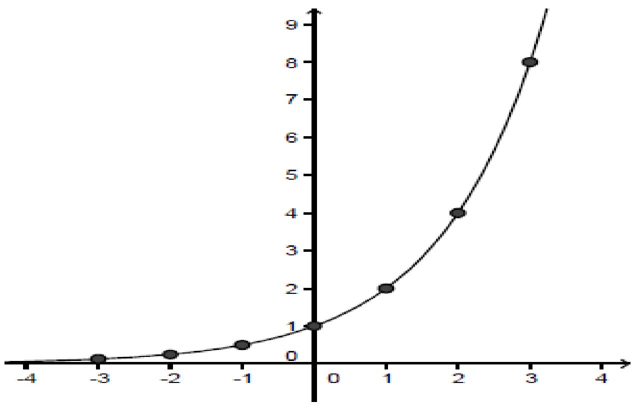
$$f(0) = 2^0 = 1, f(1) = 2^1 = 2 f(2) = 2^2 = 4 f(3) = 2^3 = 8$$

$$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2} f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4} f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Next form a table from these values.

X	-3	-2	-1	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Then, plot the corresponding points on a coordinate plane and connect them with a smooth curve for the desired graph



Let's examine some characteristics of the graph of the exponential function

$$f(x) = 2^x$$

- A vertical line will cross the graph exactly at one point. The same is true for a horizontal line. The vertical line test shows that this is indeed the graph of a function. Also note that the horizontal line test shows that the function is one-to-one.
- There is no value for x such that

$$2^x = 0$$

So the graph never touches the x-axis but it approaches the x-axis on the left and cross the y-axis at $y = 1$ i.e $(0,1)$ is y-intercept. Tothe right the function value gets larger. We say that the values grow with out bound. Then we call $y = 0(negative\ x - axis)$ thehorizontalasymptote.

Example2; draw the graph of

$f(x) = (\frac{1}{2})^x$

First choose convenient value for x, let $x = -3, -2, -1, 0, 1, 2$ and 3

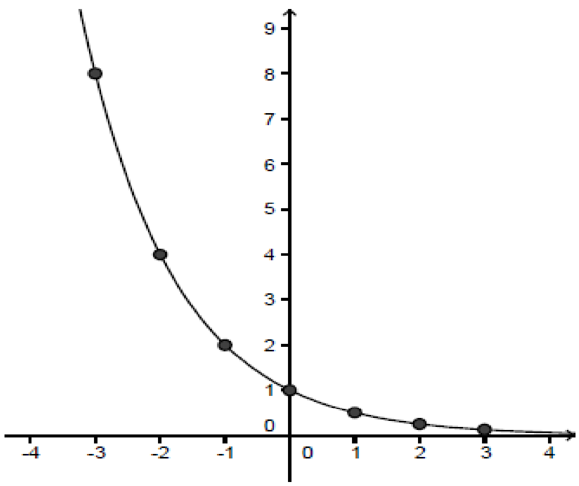
$$f(0) = (\tfrac{1}{2})^0 = 1, f(1) = (\tfrac{1}{2})^1 = \frac{1}{2} f(2) = (\tfrac{1}{2})^2 = \tfrac{1}{4} f(3) = (\tfrac{1}{2})^3 = \tfrac{1}{8}$$

$$f(-1) = (\frac{1}{2})^{-1} = 2^1 = 2 f(-2) = (\frac{1}{2})^{-2} = 2^2 = 4 f(-3) = (\frac{1}{2})^{-3} = 2^3 = 8$$

Next form a table from these values.

x	-3	-2	-1	0	1	2	3
F(x)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Then, plot the corresponding points on a coordinate plane and connect them with a smooth curve for the desired graph



Let's examine some characteristics of the graph of the exponential function

$$f(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$$

- A vertical line will cross the graph exactly at one point. The same is true for a horizontal line. The vertical line test shows that this is indeed the graph of a function. Also note that the horizontal line test shows that the function is one-to-one.
- There is no value for x such that

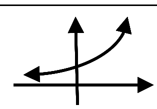
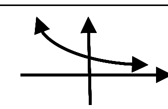
$$\left(\frac{1}{2}\right)^x = 0$$

So the graph never touches the x-axis but it approaches the x-axis on the right and cross the y-axis at y=1 $i.e$ (0,1) is y-intercept. To the left the function value gets larger. We say that the values grow without bound. Then we call $y = 0(postive\ x - axis)$ thehorizontal asymptote**.**

Note:-

1. If $a > 1$, the graph increases from left to right. If $0 < a < 1$, the graph decreases from left to right.
2. All exponential graphs will have the following in common
    - The y-intercept will be 1.
    - The graph will approach, but not touch the x-axis.
    - The graph will represent one-to-one functions.

Summery for General Properties of Exponential Function

	$\boldsymbol{f(x) = a^x}$ **for** $\boldsymbol{a} > 1$		$\boldsymbol{f(x) = a^x}$ **for 0** $< a < 1$
1	Domain is real number	1	Domain is real number
2	Range is positive real number	2	Range is positive real number
3	Has no x-intercept and (0,1) is y-intercept	3	Has no x-intercept and (0,1) is y-intercept
4	f is an increasing function	4	f has a decreasing function
5	The graph of f is more approaches to the negative x-axis as x goes to negative infinity (i.e negative x-axis is an asymptote)	5	The graph of f is more approaches to the positive x-axis as x goes to positive infinity (i.e positive x-axis is an asymptote)
6	For $x < 0, 0 < f(x) < 1$ *and* $For\ x > 0, f(x) > 1$	6	For $x < 0, f(x) > 1$ *and* $For\ x > 0, 0 < f(x) < 1$
7	If $a > b > 1, then\ a^x > b^x\ for\ x > 0,$ $a^x < b^x\ for\ x < 0$ and $a^x = b^x$ for $x = 0$	7	If $0 < a < b < 1, then\ a^x > b^x\ for$ $x > 0\ and\ a^x < b^x\ for\ x < 0$
8	General shape of the graph 	8	General shape of the graph 

The exponential function is always one-to-one. This yields an important property that can be used to solve certain types of equations involving exponents.

3.2.2.2. Solving Exponential Equation and Inequalities

$If\ a > 0\ and\ a \neq 1, x, y \in \mathbb{R}\ then,$

- $a^x = a^y if\ and\ only\ if\ x = y$
- If $a > 1$, then $a^x > a^y if\ and\ only\ if\ x > y$
- If $0 < a < 1$, then $a^x > a^y if\ and\ only\ if\ x < y$

Example1: find the value of x which satisfies each of the following equations.

a. $2^x = 64.$

Solution: $2^x = 64$

$$2^x = 2^6 \Leftrightarrow x = 6$$

b. $\sqrt[3]{128} = 4^{2x}$

Solution:− $\sqrt[3]{128} = 4^{2x}$

$$\Longrightarrow 128^{\frac{1}{3}} = (2^2)^{2x}$$

$$\Longrightarrow (2^7)^{\frac{1}{3}} = (2^2)^{2x}$$

$$\Longrightarrow 2^{\frac{7}{3}} = 2^{4x} \Leftrightarrow 4x = \frac{7}{3}$$

$$\Longrightarrow x = \frac{7}{4x3} = \frac{7}{12}$$

c. $16^{3x} = 8^{2x-1}$

Solution: $16^{3x} = 8^{2x-1}$

$$\Longrightarrow (2^4)^{3x} = (2^3)^{2x-1}$$

$$\Longrightarrow 2^{4(3x)} = 2^{3(2x-1)}$$

$$\Longrightarrow 2^{12x} = 2^{6x-3} \Leftrightarrow 12x = 6x - 3$$

$$\Leftrightarrow 12x - 6x = -3$$

$$\Leftrightarrow 12x - 6x = -3$$

$$\Leftrightarrow \ 6x = -3$$

$$\Leftrightarrow \ x = \frac{-3}{6} = \frac{-1}{2}$$

Example: Solve each of the following exponential inequalities

a. $(\frac{5}{6})^{3x-9} \leq 1$

Solution: $(\frac{5}{6})^{3x-9} \leq 1 \Longrightarrow (\frac{5}{6})^{3x-9} \leq (\frac{5}{6})^0$

$$\Leftrightarrow 3x - 9 \geq 0$$

$$\Leftrightarrow 3x \geq 9$$

$$\Leftrightarrow x \geq 3$$

b. $3^{x+2}9^x \geq 243 \Longrightarrow 3^{x+2}(3^2)^x \geq 3^5$

$$\Longrightarrow 3^{3x+2} \geq 3^5 \quad \Leftrightarrow 3x + 2 \geq 5$$

$$\Leftrightarrow 3x \geq 5 - 2$$

$$\Leftrightarrow x \geq 1$$

Exercise 3.12

1. Solve each of the following exponential equation and inequalities

a. $3^{6x+5} = 27$

b. $(\frac{4}{9})^{x}.(\frac{27}{8})^{x-1} = \frac{2}{3}$

c. $6^{3x-5} \leq 1$

d. $5^{x^2+2x} > 125$

2. Solve each of the following exponential inequalities

a. $2^{2x-1} \leq 32$

b. $4^{3x+2} > 8^{2x-1}$

3. Argue that

a. If $a > b \geq 1, then\ a^x > b^x\ for\ all\ x \in \mathbb{R}^+$.

b. If $x \geq y\ and\ a \geq 1, then\ a^x \geq b^x$.

3.3 Logarithmic Function

3.3.1, Definition of Logarithm Function

Let x and a are a positive real numbers with $a \neq 1$, then the logarithm of x to the base a is denoted by log_a^x and log_a^x gives answer to the question "for what power of 'a' gets 'x' "

Example1:

$$\log_5 25 = 2\ because\ 5^2 = 25$$

$\log_{\frac{3}{2}}\frac{4}{9} = -2, because\ (\frac{3}{2})^{-2} = \frac{4}{9}$

Definition:- Let y is any real number and $y = log_a^x$, then$y = log_a^x\ if\ and\ only\ if\ x = a^y$

Example1:

1, Write the following equations in exponential form.

a. $\log_3 81 = 4$

$$\log_3 81 = 4\ if\ and\ only\ if\ 3^4 = 81$$

b. $\log_7 7 = 1$

$$\log_7 7 = 1\ if\ and\ only\ if\ 7^1 = 7$$

c. $\log_{\frac{1}{2}}\frac{1}{8} = 3$

$$\log_{\frac{1}{2}}\frac{1}{8} = 3\ if\ and\ only\ if\ (\frac{1}{2})^3 = \frac{1}{8}$$

2, Convert each exponential statement to an equivalent logarithmic statement.

a. $2^x = 16$

$$2^x = 16 \; if \; and \; only \; if \log_2 16 = x$$

b. $3^4 = y$

$$3^4 = y \; if \; and \; only \; if \; \log_3 y = 4$$

Activity3.12

Compute each logarithm.

a. $\log_8{}^{512}$, b. $\log_6 \frac{1}{216}$ c. $\log_{10}{}^{0.00001}$ d, $\log_{\frac{1}{3}} 27$

Rules of logarithm

The following logarithm laws hold for any base $a > 0 \; and \; a \neq 1$, any positive real number x and y, and any real number n.

1. **Product Rule**

$$\log_a xy = \log_a x + \log_a y$$

Proof; let$x = a^m \Leftrightarrow \log_a x = m$ and $y = a^n \Leftrightarrow \log_a y = n$

$xy = a^m a^n = a^{m+n}$(Power rule of exponent)

$$xy = a^{m+n} \Leftrightarrow \log_a xy = m + n = \log_a x + \log_a y$$

$\therefore \log_a xy = \log_a x + \log_a y$

2. **Quotient rule**

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

Proof; exercise left for students

3. **Power Rule: -**

$$\log_a x^n = n \log_a x$$

Proof; exercise left for students

4. **Change of base**

$$\log_y x = \frac{\log_a x}{\log_a y}, \quad y \neq 1$$

Proof; Let $\log_y x = c \Leftrightarrow y^c = x$, for c is any real number.

Hence; $\log_a x = \log_a y^c$, for any positive real number $a \neq 1$

$$\log_a x = c \log_a y$$

$$c = \frac{\log_a x}{\log_a y}$$

$$\therefore\ \log_y x = \frac{\log_a x}{\log_a y}$$

5.Other rules

$\log_a 1 = 0,$ $\quad a^{\log_a x} = x \log_{a^n} x = \frac{1}{n}\log_a x$

$\log_a a = 1,$ $\quad \log_a \frac{1}{x} = -\log_a x, \log_a a^n = n$

Example1; Simplify each of the following logarithms

a. $\log_6 9 + \log_6 4 = \log_6 9\times4 = \log_6 36 = 2$

b. $\log_3 \frac{1}{81} = -\log_3 81 = -\log_3 3^4 = -4\log_3 3 = -4$

c. $\log_4 96 - \log_4 6 = \log_4 96/6 = \log_4 16 = 2$

Exercise3.13

1. Rewrite $\log_3 x + \log_3 2$ as a single logarithm.
2. Rewrite as a single logarithm.
   a. $3\log_5(x + 2) - 2\log_5(x - 1) - 2\log_5(x - 7)$
   b. $\log_2(x^2 - 16) - \log_2(x + 4)$

Logarithmic Function

To develop the idea of logarithmic function we must return to the exponential function

$$f(x) = \{(x,y): y = a^x, a > 0,\ a \neq 1\}$$

Recall that exponential function is a one to one function, since it passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a***. To find the inverse function interchanging the role of x and y, we have the inverse function

$$f^{-1}(x) = \{(x,y): x = a^y, a > 0,\ a \neq 1\} ------------(1)$$

*i.e*The logarithmic function g with base 'a' is the inverse of the functionf(x) = $a^x, a > 0, a \neq 1$. We write $g(x) = log_a x$. That is,

$$y = log_a^x \ if\ and\ only\ if\ x = a^y$$

The first equation is in logarithmic form and the second is in exponential form. In general any function defined in this form is called **Logarithmic Function**

Every logarithmic function of this form has all positive real numbers as its domain and all real numbers as its range.

Examples1;

$y = log_2^x, y = log_{\frac{1}{2}}^x,\ y = log_5^x$, etc are logarithmic function

The following are properties of an exponential function and are useful in drawing their graphs.

- If x = 1, then $f(x) = log_a^1 = 0$ for all real number $a > 0,\ \ a \neq 1$, and hence the graph of any function $y = log_a^x$ pass through the point (1,0). That is the graph of any logarithmic function cross the x-axis at (1, 0) and never cross the y-axis at all.
- If $a > 1$ then $y = log_a^x > 0$ for all $x > 1$ and in fact $y = log_a^x$ keeps increasing without bound as x increases. On the other hand for $0 < x < 1$, we get $y = log_a^x < 0$ keeps decreasing indefinitely to negative infinity as x approaching to zero and the graph approaches to the negative y-axis.
- If $0 < a < 1$, then $log_a^x < 0$ for all x > 1, and $for\ 0 < x < 1\ we\ have\ \ log_a^x > 0$. In fact $y = log_a^x$ keeps increasing without bound as x goes to zero andthe graph approaches to the positive y-axis
- The graph of $g(x)\ = log_a^x$is the reflection of the graph of y = $log_{\frac{1}{a}}^x$along the x-axis.

3.3.3. Graph of Logarithmic Function

Steps in graphing logarithmic Function

1. Establish a table of values by considering the function in the form of $y = log_a^x$
2. Plot points from the table of values on the coordinate axis
3. Connect these points with a smooth curve to form the graph.

Example1; draw the graph of the following functions and see their properties.

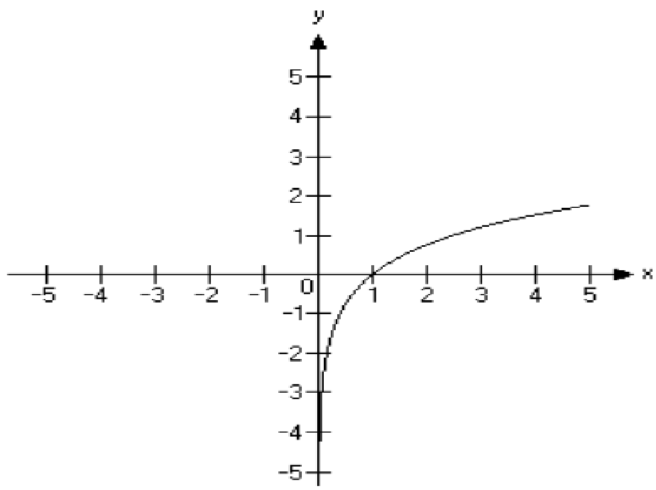
a. $y = log_2^x$ b. $y = log_{\frac{1}{2}}^x$

Solution; $y = log_2^x$

First choose convenient value for x, and summarize as in the table below.

X	1/8	1/4	½	1	2	4	8
log_2^x	-3	-2	-1	0	1	2	3

Then, plot the corresponding points on a coordinate plane and connect them with a smooth curve for the desired graph



The graph of $f(x) = \log_2^x$

Let's examine some characteristics of the graph of the exponential function

$$f(x) = log_2^x$$

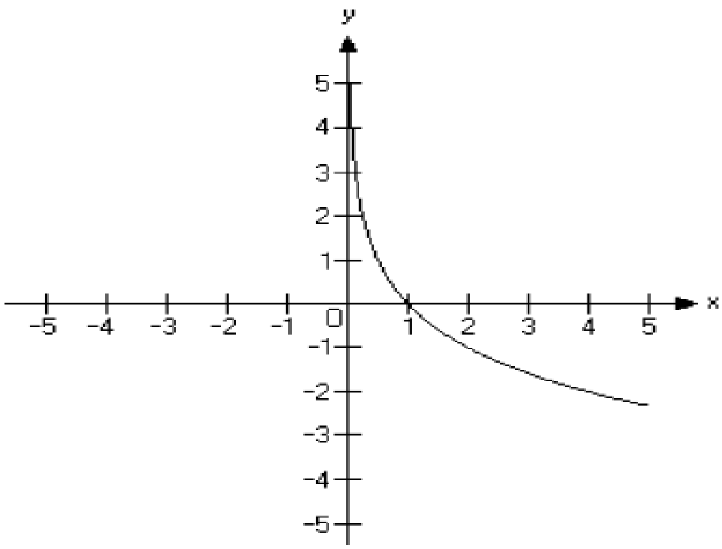
- A vertical line will cross the graph exactly at one point. The same is true for a horizontal line. The vertical line test shows that this is indeed the graph of a function. Also note that the horizontal line test shows that the function is one-to-one.
- The function is not defined at x = 0, this implies the graph never cross the y-axis, but it approaches to the negative y-axis, Then we call $x = 0(negative\ y - axis)$ the vertical asymptote and cross the x-axis at $x = 1 i.e$ (1,0) is x-intercept.

b. $y = log_{\frac{1}{2}}^x$

First choose convenient value for x, and summarize as in the table below.

x	$1/8$	$1/4$	½	1	2	4	8
$log_{\frac{1}{2}}^x$	3	2	1	0	-1	-2	-3

Then, plot the corresponding points on a coordinate plane and connect them with a smooth curve for the desired graph



The graph of $f(x) = log_{\frac{1}{2}}^{x}$

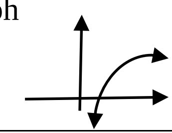
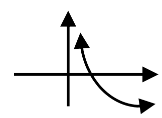
Let's examine some characteristics of the graph of the exponential function $f(x) = log_{\frac{1}{2}}^{x}$

- A vertical line will cross the graph exactly at one point. The same is true for a horizontal line. The vertical line test shows that this is indeed the graph of a function. Also note that the horizontal line test shows that the function is one-to-one.
- The function is not defined at x = 0, this implies the graph never cross the y-axis, but it approaches to the positive y-axis, Then we call $x = 0(postive\ y - axis)$ the vertical asymptote and cross the x-axis at x=1 $i.e$ (1,0) is x-intercept.

Note

1. If $a > 1$, the graph increases from left to right. If $0 < a < 1$, the graph decreases from left to right.
2. All logarithmic function graphs will have the following in common
   - The x-intercept will be 1.
   - The graph will approach, but not touch the y-axis.
   - The graph will represent one-to-one functions.

Summary for General Properties of Exponential Function

	$f(x) = \log_a^x$ for $a > 1$		$f(x) = \log_a^x$ for $0 < a < 1$
1	Domain is positive real number	1	Domain is positive real number
2	Range is real number	2	Range is real number
3	Has no y-intercept and (1,0) is x-intercept	3	Has no y-intercept and (1,0) is x-intercept
4	f is an increasing function	4	f has a decreasing function
5	The graph of f is more approaches to the negative y-axis as x goes to zero(i.e negative y-axis is an asymptote)	5	The graph of f is more approaches to the positive y-axis as x goes to zero (i.e positive y-axis is an asymptote)
6	For $0 < x < 1,\ log_a^x < 0\ and$ $For\ x > 1,\quad log_a^x > 0$	6	For $0 < x < 1, log_a^x > 0\ and$ $For\ x > 1,\quad log_a^x < 0$
7	If $b > a > 1, then\ log_a^x > log_b^x\ for\ x > 1,$ $log_a^x < log_b^x\ for\ 0 < x < 1$ and $log_a^x = log_b^x$ for $x = 1$	7	If $0 < b < a < 1, then\ log_b^x > log_a^x\ for$ $x > 1\ and\ log_b^x < log_a^x\ for\ 0 < x < 1$
8	General shape of the graph 	8	General shape of the graph 

Activity 3.13

i. Sketch the graph of each of the following functions

a. $f(x) = log_5^x$ b. $f(x) = log_{\frac{1}{3}}^x$ c. $f(x) = log_{10}^x$

ii. Draw the graphs of $f(x) = log_{\frac{1}{5}}^x$ and $f(x) = log_5^x$ on the same coordinate plane and

compare their graphs.

Note; The functions**,** $y = log_a^x$ and $y = a^x$ are inverse to each other, then their graphs are symmetric with respect to the line $y = x$.

Activity 3.14

Draw the graphs of $f(x) = log_{10}^x$ and $g(x) = 10^x$ on the same coordinate plane and compare their graphs

The logarithmic function is always one-to-one. This yields an important property that can be used to solve certain types of equations involving logarithms.

1.1.4.4. Solve Equation and inequalities involving logarithms

***Definition**:* A **logarithmic equation** is an equation that contains a logarithmic expression.

Solving Logarithmic Equations

When asked to solve a logarithmic equation, the first thing we need to decide is how to solve the problem. Some logarithmic problems are solved by simply dropping the logarithms while others are solved by rewriting the logarithmic problem in exponential form.

How do we decide what is the correct way to solve a logarithmic problem?

The key is to look at the logarithmic problem and decide if the problem contains only logarithms or if the problem contains terms without logarithms.

- If we consider the problem contains terms without logarithms. So, most probably the correct way to solve these types of logarithmic problem is to rewrite the logarithmic problem in exponential form.
- If we consider the problem contains only logarithms. So, the correct way to solve these types of logarithmic problem is to simply drop the logarithms.

When solving logarithmic equation, we may need to use the properties of logarithms to simplify the problem first.

Solving Logarithmic Equations Containing Only Logarithms

Let x and y are positive real numbers and $a > 0$ and $a \neq 1$. Then

$$\log_a x = \log_a y \ \ if \ and \ only \ if \ x = y$$

This statement says that if an equation contains only two logarithms, on opposite sides of the equal sign, with the same base then the problem can be solved by simply dropping the logarithms.

Steps for Solving Logarithmic Equations Containing Only Logarithms

Step1. Determine the domain of the problem.

Step2. Use the properties of logarithms to simplify the problem if needed.

Step3. Rewrite the problem without the logarithms.

Step4. Simplify the problem.

Step5. Solve for x.

Step6. Check your answer(s).

Remember we cannot take the logarithm of a negative number, so we need to make sure that when we plug our answer(s) back into the original equation we get a positive number. Otherwise, we must drop that answer(s).

Steps for Solving Logarithmic Equations Containing Terms without Logarithms

Step1. Determine the domain of the problem.

Step2. Use the properties of logarithms to simplify the problem if needed.

Step3. Rewrite the problem in exponential form.

Step4. Simplify the problem.

Step5. Solve for x.

Step6. Check your answer(s).

Example1; solve each of the following

a. $\log_3(x+12) - \log_3(x-3) = \log_3 6$

Solution; $Domain = \ x+12 > 0 \ and \ x-3 > 0$

$$= x > -12 \ and \ x > 3$$

$\therefore$ Domain of the problem i = $\{x : \ x \ > \ 3\}$

Now,$\log_3(x+12) - \log_3(x-3) = \log_3 6$

$$\log_3 \frac{x+12}{x-3} = \log_3 6$$

$$\frac{x+12}{x-3} = 6$$

$$x+12 = 6(x-3)$$

$$x+12 = 6x-18$$

$$6x - x = 12+18$$

$$5x = 30$$

$$x = 6$$

Check; $\log_3(x+12) - \log_3(x-3) = \log_3 6$

$$\log_3(6+12) - \log_3(6-3) = \log_3 6$$

$$\log_3 18 - \log_3 3 = \log_3 6$$

$$\log_3 \frac{18}{3} = \log_3 6$$

$$\log_3 6 = \log_3 6\textbf{(ok)}$$

b. $\log_2(x+2) + \log_2(x-5) = 3$

Solution: $Domain = \ x + 2 > 0 \ and \ x - 5 > 0$

$$= x > -2 \ and \ x > 5$$

$\therefore$ Domain of the problem $= \{x : \ x \ > \ 5\}$

Now, $\log_2(x + 2) + \log_2(x - 5) = 3$

$$\log_2(x + 2)(x - 5) = 3 \Leftrightarrow (x + 2)(x - 5) = 2^3$$

$$x^2 - 3x - 10 = 8$$

$$x^2 - 3x - 18 = 0$$

$$x^2 - 6x + 3x - 18 = 0$$

$$(x - 6)(x + 3) = 0$$

$$x = 6 \ or \ x = -3$$

$x = -3$ is not in the domain, so it is not the solution

Let us check for $x = 6$

$$\log_2(x + 2) + \log_2(x - 5) = 3$$

$$\log_2(6 + 2) + \log_2(6 - 5) = 3$$

$$\log_2 8 + \log_2 1 = 3$$

$3 + 0 = 3$**(ok)**

Logarithmic inequalities

Let x and y are positive real numbers and $a > 0 \ and \ a \neq 1$. Then

- $\log_a x < \log_a y \ \ if \ and \ only \ if \ \ x < y, \ \ for \ a > 1.$
- $\log_a x < \log_a y \ \ if \ and \ only \ if \ \ x > y, \ \ for 0 < a < 1.$
- $\log_a x = \log_a y \ \ if \ and \ only \ if \ \ x = y, for \ any a > 0 \ and \ a \neq 1$

Example; If $f(x) = \log_5 2x - 1$, then

i. State the domain

ii. Determine the value of x, if

   a. $f(x) \geq 3$

   b. $f(x) < 1$

Solution;

i. Domain $= 2x - 1 > 0 = x > \frac{1}{2}$

   Domain = $\{x : \ x > \frac{1}{2}\}$

ii. a. $f(x) \geq 3 \Longrightarrow \log_5 2x - 1 \geq 3$

$$\Rightarrow \log_5 2x - 1 \geq 3\log_5 5$$

$$\Rightarrow \log_5 2x - 1 \geq \log_5 125$$

Since $a = 5 > 1$,

$$2x - 1 \geq 125$$

$$2x \geq 126$$

$$x \geq 63$$

$b.\ f(x) < 1$ (**Exercise**)

Exercise 3.14

iii. Solve each of the following logarithmic equations and inequalities

a. $1 + 2\log_4(x + 1) = 2\log_2 x$

b. $(\log(x))^2 = 2\log(x) + 15$

c. $x log(x + 1) \geq x$, for $x \neq 0$

iv. Solve each of the following logarithmic equation and inequalities

a. $\log_6^{(x+4)} + \log_6^{(3-x)} = 1$

b. $\log(x + 5) + \log(x - 2) = \log(4 - 2x)$

c. $\log_5(x + 3) \text{ x} \log_{(x+3)}(2\text{x} + 1) \text{ x} \log_{(2x+1)} 3x = 1$

d. $\log_2(x + 3) = 3 + \log_2(6 - x)$

e. $\log_3^{(3x-5)} > \log_3^{(x+7)}$

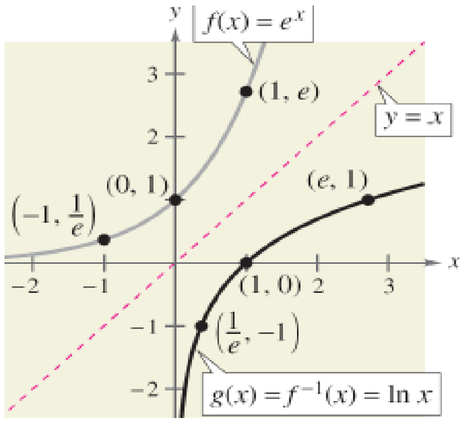
f. $\log_2^{2x} < \log_4^{(x+3)}$

g. $\log_2 x < 3 + 2\log_x 2$

The Natural Logarithmic Function

f(x) = e^x is one-to-one **natural exponential function** and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol ln*x*, read as "the natural log of *x*". Note that the natural logarithm is written without a base. The base is understood to be e.

Because the functions given by f (x) = e^x and *g*(*x*) = l n *x* are inverse functions of each other, their graphs are reflections of each other in the line ***y = x*.**



Reflection of the graph of f (x) = e^xabout the line y = x

Common logarithms

If $x > 0$, then the common logarithm of x is the logarithm of x to the base 10 and denoted by log_{10}^{x}. For convenience, the base of the common logarithm is not often written.

i.e for each $x > 0$, log_{10}^{x} is simply written as $\log x$

Before we discuss some rules for finding common logarithm of numbers, we note that any positive number '*x*' can be expressed in the form;

$$x = a \text{ x } 10^n, where \;\; 1 \leq a < 10 \; and \; 'n' \; is \; any \; integer$$

This form of a number ***x***is known as the **scientific notation (standard form)** of the number.

Now to find the common logarithm of a number '*x*';

$$\log x = \log a \text{ x } 10^n$$

$$= \log a + \log 10^n$$

$$= \log a + n \log 10$$

$$\log x = \log a + n$$

Where $0 \leq \log a < 1$ *is called* **mantissa** *of* $\log x$ and 'n' is called **characteristics** of $\log x$

∴ For any positive real number x, its common logarithms can be written as;

$$\mathbf{\log x = Mantissa + Characterstics}$$

We can see that any positive number different from one can be base of a logarithm. It is imperial to prepare a logarithm table for each base. However the change of base rule has reduced our problem of constructing a table for each possible base. So it is sufficient to construct a logarithm table only for common logarithm to determine the logarithm of any number to any base.

In general the logarithm of a number between 1 and 10 is a decimal between 0 and 1 and find this value from the common logarithm table. The common logarithm table is a table of mantissa.

The mantissa of log_{10}^{x} can be given to any decimal place but it is often given only a decimal number of four decimal place.

Example; find the common logarithm of

a. 26300 b. 0.0002630, given that $\log 2.63 = 0.4200$

Solution:

$a.\ 26300 = 2.63\ x\ 10^4$, then

$$\log 26300 = \log 2.63 \text{ x } 10^4$$
$$= \log 2.63 + \log 10^4$$
$$= \log 2.63 + 4 \log 10$$
$$\log 26300 = 0.4200 + 4 = 4.4200$$

$$b.\ 0.0002630 = 2.63\ x\ 10^{-4}$$
$$\log 0.0002630 = \log 2.63 \text{ x } 10^{-4}$$
$$= \log 2.63 + \log 10^{-4}$$
$$= \log 2.63 - 4 \log 10$$
$$\log 0.0002630 = 0.4200 - 4 = -3.5800$$

All the above numbers we considered at most three significant digits. Suppose let as find the logarithm of 3485.

$$3485 = 3.485\ x\ 10^3$$
$$\log 3485 = \log(3.485\ x\ 10^3)$$
$$= \log 3.485 + \log 10^3$$
$$= \log 3.485 + 3\log 10$$
$$= \log 3.485 + 3$$

$\log 3.485$ Can't found from the logarithm table but

$$\left.\begin{array}{r} \log 3.48 = 0.5416 \\ \log 3.485 = ? \\ \log 3.49 = 0.5428 \end{array}\right\}$$

Difference $(0.5428 - 0.5416 = 0.0012)$

Since 3.485 is mid way between 3.48 and 3.49, then

$$\log 3.485 \approx 0.5416 + \frac{1}{2}(0.0012) \approx 0.5422$$

Example1; find the mantissa and characteristics of the following

a. $\log x = 5.6234$ b. $\log x = -2.4167$

Solution;

$$a.\ \log x = 5.6234$$
$$= 0.6234 + 5$$

∴ Mantissa of $\log x$ is 0.6234 and characteristics of $\log x$ is 5

b. $\log x = -2.4167$

$$= -2 + (-0.4167)$$
$$= -2 - 1 + 1 - 0.4167$$
$$= -3 + 0.5833$$

∴ Mantissa of $\log x$ is 0.5833 and characteristics of $\log x$ is -3

Example2; find the common logarithm of the following expression

a. $\frac{(23.2)^5 x296}{42.2}$ b. $\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}}$

Solution;

a. $\frac{(23.2)^5 x296}{42.2}$

$$\log\frac{(23.2)^5 x296}{42.2} = \log((23.2)^5\, x296) - \log 42.2$$
$$= \log(23.2)^5 + \log 296 - \log 42.2$$
$$= 5\log(2.32\,x10) + \log 2.96x10^2 - \log 4.22x10$$
$$= 5[\log 2.32 + \log 10] + [\log 2.96 + \log 10^2] - [\log 4.22 + \log 10]$$
$$= 5[0.3655 + 1] + [0.4713 + 2] - [0.6253 + 1]$$
$$= 6.8275 + 2.4713 - 1.6253$$
$$= 7.6735$$

b. $\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}}$

$$\log\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}} = ?\ (\boldsymbol{Exercise})$$

Antilogarithm

Antilogarithm is the inverse function oflogarithm,and defined as

$$antilog_a(\log_a y) = y = \log_a(antilog_a^y)$$

The antilogarithm in base ‘a’of ‘y’is therefore a^y

$$antilog_a^y = x \Longrightarrow a^y = x$$

If x = $\log^y$, then y is the antilogarithm of x.

If $\log x = Mantissa + Characterstics$, then

$$x = antilog(Mantissa + Characterstics)$$

$$\boldsymbol{x = antilog(mantissa)\text{x } antilog(characterstics)}$$

To find the antilogarithm of a number, we must properly use the role of mantissa and characteristics.

To find the antilogarithm of a number

- Find the mantissa in the body of the common logarithm table
- Read the number whose common logarithm is this mantissa
- Then, by the proper use of the characteristics fix the decimal place

Example1; find the anilogarithm of 4.2833

Solution: let x is antilogarithm of 4.2833

$$\log x = 4.2833 = 0.2833 + 4$$

$$antilog(\log x) = antilog(0.2833 + 4)$$

$$x = antilog(0.2833)\text{x antilog}(4)$$

From the common logarithm table $antilog(0.2833) = 1.92$ and $\text{antilog}(4) = 10^4$. So

$$x = antilog(0.2833)\text{x antilog}(4)$$

$$= 1.92 \text{ x } 10^4$$

$$= 19{,}200$$

Example2; find the antilog of -6.4647.

Solution; (excrsise)

Computition using logarithms

Steps to compute any numerical expresion using the concepts of logarithm;

1. Find the logarithm of the exprssion
2. Write the result obtaind in step one as the sum of mantissa and characterstics.
3. Find the antilogarithm of the result obtaind in step two.
4. The result obtained in step three is the value that we want.

Example3; simplify each of the following using the concept of logarithm.

a. 36.275 x 123.514

b. $\frac{658\ x\ 0.00345}{845}$

c. $\frac{\sqrt[3]{563\ x 4.52}}{86400}$

d. $\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}}$

Solution; simplify only d, the rest are exercise for the reader.

$d.\ \frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}}$

$$\log\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}} = \log(246)^3\sqrt{52.34} - \log\sqrt[3]{56.23}$$

$$= \log(246)^3 + \log(52.34)^{\frac{1}{2}} - \log(56.23)^{\frac{1}{3}}$$

$$= 3\log(2.46x10^2) + \frac{1}{2}(\log 5.234x10) - \frac{1}{3}(\log 5.623x10)$$

$$= 3[\log 2.46 + \log 10^2] + \frac{1}{2}[\log 5.23 + \log 10] - \frac{1}{3}[\log 5.62 + \log 10]$$

$$= 3[0.3909 + 2] + \frac{1}{2}[0.7185 + 1] - \frac{1}{3}[0.7497 + 1]$$

$$= 7.1727 + 0.8593 - 0.5832$$

$$= 7.4488$$

$$\log\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}} = 7.4488 = 7 + 0.4488$$

$$\frac{(246)^3\sqrt{52.34}}{\sqrt[3]{56.23}} = antilog(7 + 0.4488)$$

$$= antilog7 + antilog\ (0.4488)$$

$$= 10^7\text{x } 2.81$$

$$= 28{,}100{,}000$$

Activity 1.15

1. Find the common logarithm of the following number

   a. 564,421,000 b. $\frac{658\ x\ 0.00345}{845}$ $C.$ 564,421,000 d. $\frac{\sqrt[5]{0.243}x\sqrt{24.79}}{(0.814)^3}$

2. Find the antilogarithm of the given number

   a. 4.6542 b. -0.5490 c. 6.6542 d. -3.5490

3. Use common logarithm table and solve for x, if $\log x = 2.0150$
4. If $\log 2.54 = 0.4048$, without using common logarithm table find;

   a. $\log 25400$ b. $\log 0.00254$ c. $\log 254$
5. Use common logarithm table to solve for x,

a. $\log x = -2.5952$ b. $\log_{201} 404 = x$

6. If $\log 3.12 = 0.4942$, without using common logarithm table find;

   a. $\log 31200$ b. $\log 0.00312$

1.1.4.5. Application of Exponential and Logarithmic Function

i. An Interest Application

If an investment of P Birr earns interest at an annual interest rate 'r'and the interest is compounded 'n'times per year, then the amount in the account after't'years is given by

$$A = p\left(1 + \frac{r}{n}\right)^{nt}$$

Example; If Birr 1000 is placed in an account with an annual interest rate of 6%, find out how long it will take the money to double when interest is compounded **annually** and **quarterly.**

Solution; given, p = 1000, n =1 r = 6% =0.06 and A = 2x1000 = 2000.

We want to find the time (t =?)

- Compounding interest annually.

$$A = p\left(1 + \frac{r}{n}\right)^{nt}$$

$$2000 = 1000\left(1 + \frac{0.06}{1}\right)^{t}$$

$$(1.06)^t = \frac{2000}{1000} = 2$$

$$(1.06)^t = 2 \Leftrightarrow t = \log_{1.06} 2 = \frac{\log 2}{\log 1.06} = \frac{0.301}{0.0253} = 11.897 \approx 12$$

∴ After 12 year the money becomes double

- Compounding interest quarterly.**(Exercise)**

ii. A decay Application

Example 2;Aradioactive substance of original weight 100grams, which reduced by 5% each year, then find the weight of the substance after 20 years.

Solution: let W_n is the weight of the substance after nth year; then

$$W_n = 100(1 - 0.05)^n = 100(0.95)^n$$

If n = 20, then

$$W_{20} = 100(0.95)^{20}$$

$\log W_{20} = \log 100(0.95)^{20} = \log 100 + \log(0.95)^{20} = 2 + 20\log 9.5\text{x}10^{-1}$

$$= 2 + (\log 9.5 + \log 10^{-1}) = 2 + 20(-1 + 0.9777) = 1..5530$$

$$W_{20} = antilog(1.5530) = antilog(1 + 0.5530) = antilog(1)\text{xantilog}(0.5530)$$

$$= 10\text{x}3.57 = 35.7$$

∴ After 20 years the substance has 35.7grams remains

iii. A Population Application

Example1; A town's population is presently 10,000. Given a projected growth rate of 7% per year,*t* years from now the population *P* will be given by

$$P = 10{,}000e^{0.07t}$$

In how many years will the town's population double?

Solution; given P = double = 2x10, 000 =20,000, t = ?

$$P = 10{,}000e^{0.07t}$$

$$20{,}000 = 10{,}000e^{0.07t}$$

$$e^{0.07t} = 2$$

$$0.07t = \ln 2$$

$$t = \frac{ln2}{0.07} = 9.9$$

The population will double in approximately 9.9 years.

iv. Sound Intensity

The response of the human ear to sound waves follows closely to a logarithmic function of the form $R = k \log I$, where *R* is the response to a sound that has an intensity of *I*, and *k* is a constant of proportionality. Thus, we define the relative sound intensity level

$$\text{SL} = 10\log\frac{I}{I_0}$$

The unit of *SL*is called the decibel (abbreviated dB). *I* is the intensity of the soundexpressed in watts per meter and $I_o$is the reference intensity defined to be 10^{-12} w/m^2.

Example; If we measured a sound intensity to be 560 greater that the threshold reference, what would be the sound level expressed in dB?

Solution;

$$\text{SL} = 10\log\frac{I}{I_0}$$

$$SL = 10\log\frac{560x10^{-12}}{10^{-12}} = 10\log(5.60\ x\ 10^2) = 10[\log 5.60 + 2]$$

$$= 10[0.7482 + 2] = 10[2.7482] = 27.48dB$$

Example; The threshold of pain is about 120 dB. How many times greater in intensity (in w/m^2) is this?**(Exercise)**

v. The pH of a Solution

The pH of a solution is a measure of the **acidity of the solution**. It is defined as:

$$\boldsymbol{PH = -\log[H_3O^+]}$$

Where; $[H_3O^+]$ is the **concentration** of **hydronium ions** in the **solution**.

Example1; Calculate the pH of a solution, if the concentration of hydronium ionsin the solution is 0.0125M.

Solution:$H_3O^+ = 0.0125M$, then

$$PH = -\log[H_3O^+] = -\log[0.0125] = -\log[1.25x10^{-2}] = -[-2 + \log 1.25]$$

$$= -[-2 + 0.0969] = -[-1.9031] = 1.9031$$

Exercise 3.15

1. If Birr 5000 is placed in a an account with an annual interest rate of 9%, how long will it take the amount to double if the interest is
   a. Compounded annually
   b. Compounded semi annually
   c. Compounded quarterly
   d. Compounded monthly
2. The radioactive element strontium 90 has a half-life of approximately 28 years. If A_0 is the initial amount of the element, then the amount A remaining after t years is given by

$$A(t) = A_0(\frac{1}{2})^{\frac{t}{28}}$$

   a. If the initial amount of the element is 100g, in how many years will 60g remains?
   b. In how many years will 75% of the original amount remains?(hint let $A = 0.75A_0$)
3. The decibel (dB) rating for the loudness of a sound is given by

$$L = 10\log\frac{I}{I_0}$$

   Where I is the intensity of that sound in watts per square centimeter and I_0 is the intensity of the “threshold” sound$I_0 = 10^{-16}W/cm^2$. Find the decibel rating of

a. A table saw in operating with intensity $I = 10^{-6} W/cm^2$

b. The sound of a passing car horn with intensity $I = 10^{-8} W/cm^2$

❖ Summary

❖ Let a and b are any real numbers, m and n are positive integers;

- $\left.\begin{array}{l} a^m x a^n = a^{m+n} \\ (ab)^n = a^n x b^n \end{array}\right\}$ Product rule
- $\left.\begin{array}{l} \frac{a^n}{b^n} = (\frac{a}{b})^n,\ b \neq 0 \\ \frac{a^n}{a^m} = a^{n-m},\ a \neq 0 \end{array}\right\}$ Quetient rule
- $(a^n)^m = a^{mn}$ Power rule
- $a^{-n} = \frac{1}{a^n} \Leftrightarrow a^n = \frac{1}{a^{-n}}$ Negative exponent
- $a^0 = 1, a \neq 0$ Zero exponent
- $(a)^{\frac{1}{n}} = \sqrt[n]{a},\ when\ \sqrt[n]{a}\ is\ a\ real\ number, n > 1$
- $(a)^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

❖ The principal n^{th} root of a is denoted by $\sqrt[n]{a}$ and defined as

- The positive n^{th} root of a if a>0 and n is even
- The negative n^{th} root of a, if a<0 and n is odd.
- 0, if a=0

❖ If a and b are real numbers and n and m are positive integer greater than 1, then

- $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, if n is even then a and b are non-negative real numbers
- $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},\ b \neq o$,if n is even then a and b are non-negative real numbers.
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$,if nm is even then a is non-negative real number.

❖ Exponential function is a function of the form

$f(x) = a^x, a > 0, a \neq 1$,The number '$a$' is called the **base** of the exponential function.

Domain of logarithmic function is positive real number and its range is real number.

- If $a > 1$ the function is an increasing function and if $0 < a < 1$ the function is a decreasing function.
- The graph of g(x) = $(\frac{1}{a})^x$is the reflection of the graph of y = a^xalong the y-axis.
- All exponential graphs will have the following in common

- The y-intercept will be 1 has no x-intercept.
- The graph will approach, but not touch the x-axis.
- The graph will represent one-to-one functions.

  - If $a > 1$, $then\ 0 < f(x) < 1\ for\ x < 0\ and\ f(x) > 1, for\ x > 0,$
  - If $0 < a < 1$, then $f(x) > 1\ for\ all\ x < 0\ and\ 0 < f(x) < 1\ For\ x > 0$

- $If\ a > 0\ and\ a \neq 1, x, y \in \mathbb{R}\ then,$
  - $a^x = a^y if\ and\ only\ if\ x = y$
  - If $a > 1$, then $a^x > a^y if\ and\ only\ if\ x > y$
  - If $0 < a < 1$, then $a^x > a^y if\ and\ only\ if\ x < y$
- Let x and a are a positive real numbers with $a \neq 1$, then the logarithm of x to the base a is denoted by log_a^x and givesanswer to the question "for what power of 'a' gets 'x' "
- For any base $a > 0\ and\ a \neq 1$, any positive real number *x* and *y*, and any real number *n*.
  - $\log_a xy = \log_a x + \log_a y$ Product rule
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$ Quetent rule
  - $\log_a x^n = n \log_a x$ Power rule
  - $\log_y x = \frac{\log_a x}{\log_a y}$, $y \neq 1$ Change of base
- Logarithmic function is a function of the form

$$f(x) = \log_a^x, a > 0 \text{ and } a \neq 1$$

  - $y = log_a^x\ if\ and\ only\ if\ x = a^y$

Domain of logarithmic function is positive real number and its range is real number.

  - The graph of any logarithmic function cross the x-axis at (1, 0) and never cross the y-axis at all.
  - If $a > 1$ then $y = log_a^x$ is always an increasing function, whereas, if $0 < a < 1$, then $y = log_a^x$ is always a decreasing function
  - The graph of $g(x) = log_a^x$ and y = $log_{\frac{1}{a}}^x$ are symmetric about the x-axis.

- If $b > a > 1, then\ log_a^x > log_b^x\ for\ x > 1,\ log_a^x < log_b^x\ for\ 0 < x < 1$ and $log_a^x = log_b^x$for$x = 1$
- If $0 < b < a < 1, then\ log_b^x > log_a^x\ for\ x > 1\ and\ log_b^x < log_a^x\ for\ 0 < x < 1$
- Let x and y are positive real numbers and $a > 0\ and\ a \neq 1$. Then
  - $\log_a x < \log_a y\ if\ and\ only\ if\ x < y,\ for\ a > 1.$

- $\log_a x < \log_a y$ *if and only if* $x > y$, *for* $0 < a < 1$.
- $\log_a x = \log_a y$ *if and only if* $x = y$, *for any* $a > 0$ *and* $a \neq 1$

- If $x > 0$, then the common logarithm of x is the logarithm of x to the base 10. For convenience, the base of the common logarithm is not often written.
- f(x) = $e^x$is one-to-one natural exponential function and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol ln*x*,
- **Antilogarithm** is the inverse function of logarithm, and defined as

$$antilog_a(\log_a y)=y = \log_a(antilog_a^y)$$

- Steps to compute any numerical expresion using the concepts of logarithm;
  - Find the logarithm of the exprssion
  - Write the result obtaind in step one as the sum of mantissa and characterstics.
  - Find the antilogarithm of the result obtaind in step two.
  - The result obtained in step three is the value that we want.

1.1.5 Power Function

Definition: A power function is a function which can be a written in the form of$f(x) = ax^r$, *where* r is a rational number (constant) and $a \in \Re$, is a fixed111 number.

Note: Don't confuse power function with exponential functions.

- ✓ **Exponential function: -**$y = a^x$(a fixed base is raised to a variable exponent).
- ✓ **Power function: -** $y = ax^r$(a variable base is raised to a fixed exponent).

Example1: - which of thefollowing are power function and which are not?

✓ **a).** $f(x) = 5x^2 + 1$ **b).** $f(x) = 5x^{\frac{-3}{4}}$ **c).** $f(x) = \sqrt[5]{\frac{1}{x}}$ **D).** $f(x) = \sqrt[3]{\frac{1}{2x}}$ **E).** $f(x) = x^{-2}$ **F).**$f(x) = 2x^{-\frac{4}{5}}$

Solution: d) power function written in the form of$f(x) = ax^r$

Thus, $f(x) = \sqrt[3]{\frac{1}{2x}} = (\frac{1}{2x})^{\frac{1}{3}} = (\sqrt[3]{\frac{1}{2}})x^{\frac{1}{3}}$this is power function which$a = \sqrt[3]{\frac{1}{2}}$

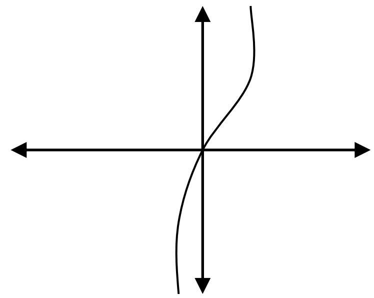
b). $f(x) = \frac{x^2}{2} = f(x) = \frac{1}{2}x^2$is a power function which $a = \frac{1}{2}$.

c). $f(x) = 2^x$is not power function(it is exponential function).

Excersice for the studentsthe rest all the above.

The Graph of Power Function

The following fingers give you some of the various possible graphs of power function with rational expression.

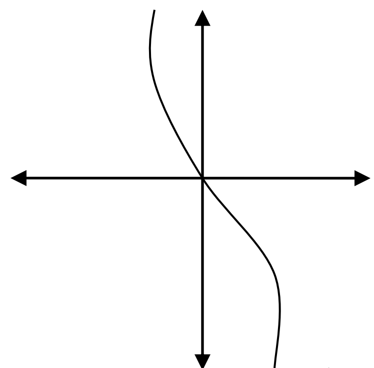


1. $\boldsymbol{f(x) = ax^{r}}$, **r is odd and** $\boldsymbol{r} > 0$

   i. If $\boldsymbol{a} > 0$

   - Domain $= \Re$
   - Range $= \Re$
   - Odd function
   - Increasing

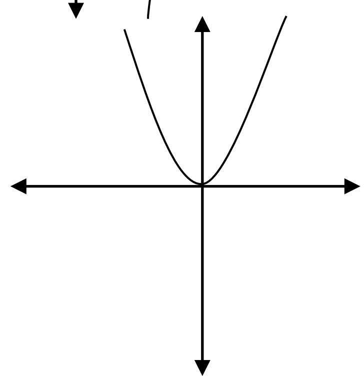
Example: $\boldsymbol{f(x) = x^{3}, x^{5}, x^{7}, ---}$



   ii. **If** $\boldsymbol{a} < 0$

   - Domain $= \Re$
   - Range$= \Re$
   - Odd function
   - Increasing

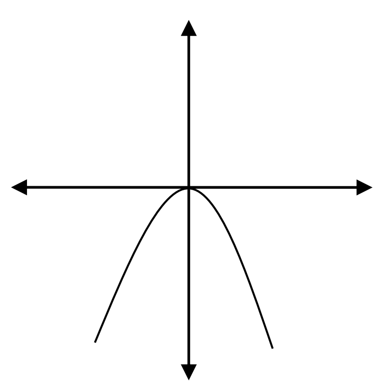
Example: $\boldsymbol{f(x) = -x^{3}, -x^{5}, -x^{7}, ---}$



2. $\boldsymbol{f(x) = ax^{r}}$, **r is an even and** $\boldsymbol{r} > 0$

   i. **If** $\boldsymbol{a} > 0$

   - Domain $= \Re$
   - Range$= [0, \infty)$
   - Even function

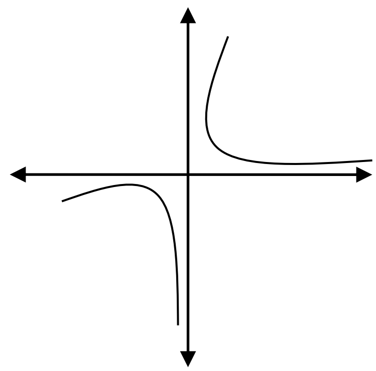


   ii. **If** $\boldsymbol{a} < 0$

   - Domain $= \Re$
   - Range$= (-\infty, 0)$

- Even function

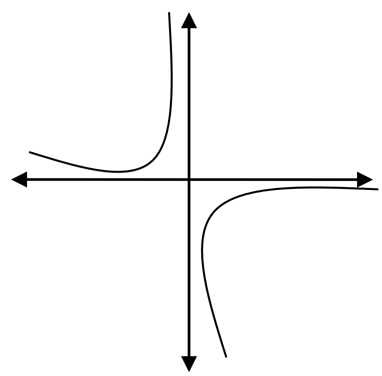
3. $\boldsymbol{f(x) = ax^r}$**, r is odd and** $\boldsymbol{r} < 0$



i. **If** $\boldsymbol{a} > 0$

- Domain = $\Re/\{0\}$
- Range= $\Re/\{0\}$)
- Odd function
- Decreasing function

Example :-$\boldsymbol{f(x) = x^{-3}, x^{-5}, x^{-7}, ---}$

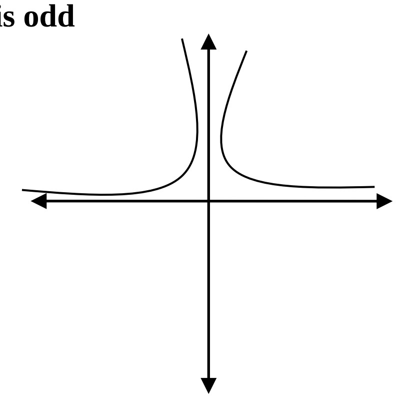


ii. **If** $\boldsymbol{a} < 0$

- Domain = $\Re/\{0\}$
- Range= $\Re/\{0\}$)
- Odd function
- Increasing function

Example :-$\boldsymbol{f(x) = -x^{-3}, -x^{-5}, -x^{-7}, ---}$

4. $\boldsymbol{f(x) = ax^{\frac{n}{m}}}$**, n is even and** $\boldsymbol{n} < 0$ and **m is odd**



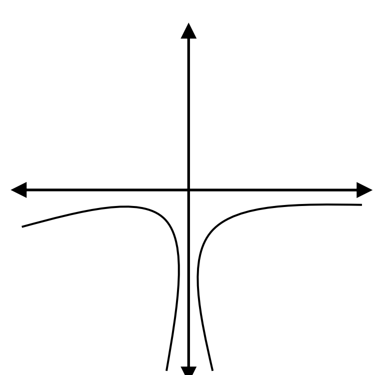
ii. **If** $\boldsymbol{a} > 0$

- Domain = $\Re/\{0\}$
- Range= $\Re/\{y: y > 0\}$
- even function

Example:-$\boldsymbol{f(x) = x^{-\frac{2}{3}}, x^{\frac{-4}{3}}, x^{\frac{-6}{5}}, ---}$

iii. **If** $\boldsymbol{a} < 0$

- Domain = $\Re/\{0\}$
- Range= $\Re/\{y: y < 0\}$

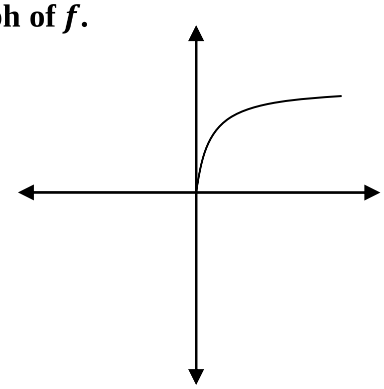


- even function

Example:-$\boldsymbol{f(x) = -x^{-\frac{2}{3}}, -x^{\frac{-4}{3}}, -x^{\frac{-6}{5}}, ---}$

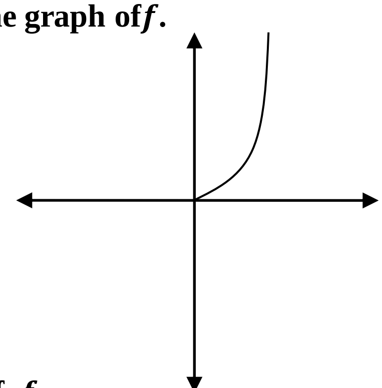
5. $\boldsymbol{f(x) = x^{\frac{n}{m}}}$**, n is even and** m **is odd, then the graph of** $\boldsymbol{f}$**.**

   - Not Symmetric
   - Neither even nor odd
   - Domain = $[0, \infty)$
   - Range= $[0, \infty)$



$\frac{m}{n} = \frac{1}{2}, \frac{3}{4}, -----$**n is even and m is odd**

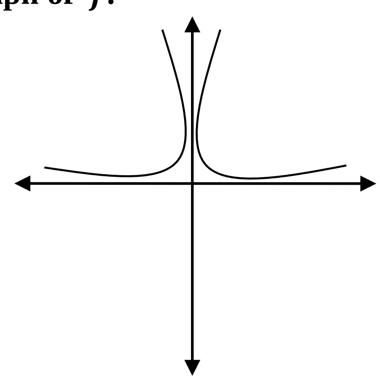
$\boldsymbol{f(x) = x^{\frac{n}{m}}}$**,** $m > n$ **n is even and** m **is odd,then the graph of**$\boldsymbol{f}$**.**



$\frac{m}{n} = \frac{3}{2}, \frac{5}{4}, \frac{7}{2}, -----$**n is even and m is odd**

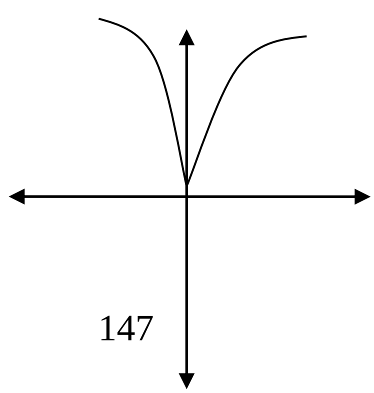
6. $\boldsymbol{f(x) = x^{\frac{n}{m}}}$**, n is oddand** m **is Even, then the graph of** $\boldsymbol{f}$**.**

   - Symmetric w.r.t Y-axis
   - even function
   - Has domain $real\ number$
   - Has range= $0 \leq y < \infty$

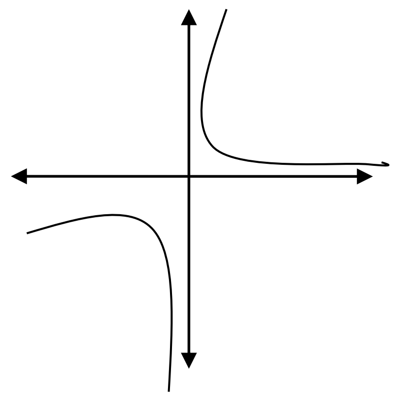


Example:- $\boldsymbol{f(x) = x^{\frac{2}{3}}, x^{\frac{4}{5}}}$

$\boldsymbol{f(x) = x^{\frac{m}{n}}}$, $\frac{m}{n} < 0$



Example:- $\boldsymbol{f(x) = x^{\frac{-2}{3}}}$



7. $\boldsymbol{f(x) = x^{\frac{-1}{n}}}$**, n is odd, then the graph of** $\boldsymbol{f}$**.**

- Symmetric
- Odd function
- Increasing function.
- Domain $= \Re/\{0\}$

Example: - $\boldsymbol{f(x) = x^{\frac{-1}{3}}}$**,** $x^{\frac{-1}{5}}, x^{\frac{-1}{7}}$

N.B all power Function of the form$\boldsymbol{ax^r}$ **with** $\boldsymbol{a = 1}$**satisfies multiplication property of**

$$\boldsymbol{f(xy) = f(x).f(y)}$$

Example: which of the following power function does not satisfies the condition

$$f(xy) = f(x).f(y)$$

a). $f(x) = \sqrt{\frac{1}{x}}$ b). $f(x) = x^3$

c). $f(x) = x^{\frac{2}{3}}$ $d). f(x) = 2x^4$

Solution: a),.$f(xy) = \sqrt{\frac{1}{xy}} = \sqrt{\frac{1}{x}}.\sqrt{\frac{1}{y}} = f(x).f(y)$

c. $f(xy) = (xy)^3 = x^3y^3 = f(x).f(y)$
d. $.f(xy) = 2(xy)^4 = 2x^4y^4$

But $f(x).f(y) = 2x^4.2y^4 = 4x^4y^4$

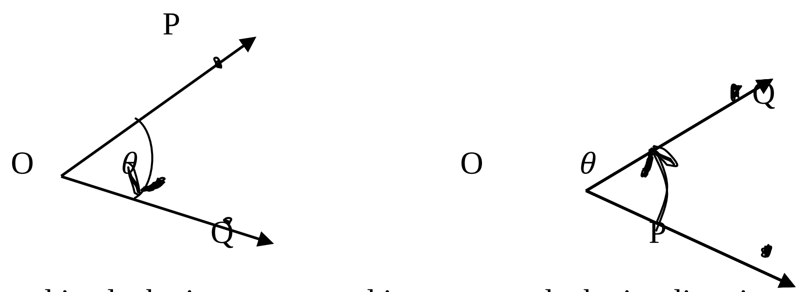
Hence $f(xy) \neq f(x).f(y)$

3.4 Trigonometric Function and their graphs

Basic terminologies

Given any $\overrightarrow{op}$ it rotate about the point O to a new position either clockwise or counter clockwise direction, it forms an angle say $\theta, then$

b. The initial position of the ray is called **initial side of the angle** $\boldsymbol{\theta}$
c. The terminal position of the ray is called the **terminal side of the angle** $\boldsymbol{\theta}$



ray rotated in clockwise direction

ray rotated in counter- clockwise direction

Positive and Negative angles

- An angle formed by rotating the ray in counter-clock wise is considered as a positive angle
- An angle formed by rotating the ray in clock-wise direction is considered as a negative angle

Example: - Identify whether each of the following angles are positive or negative and indicate the initial and terminal sides.

Angle α is negative angle initial side is $\overrightarrow{OP}$ and terminal side is $\overrightarrow{(OQ)}$

Angle θ is positive angle initial side is $\overrightarrow{(ON)}$ terminal side is $\overrightarrow{(OM)}$

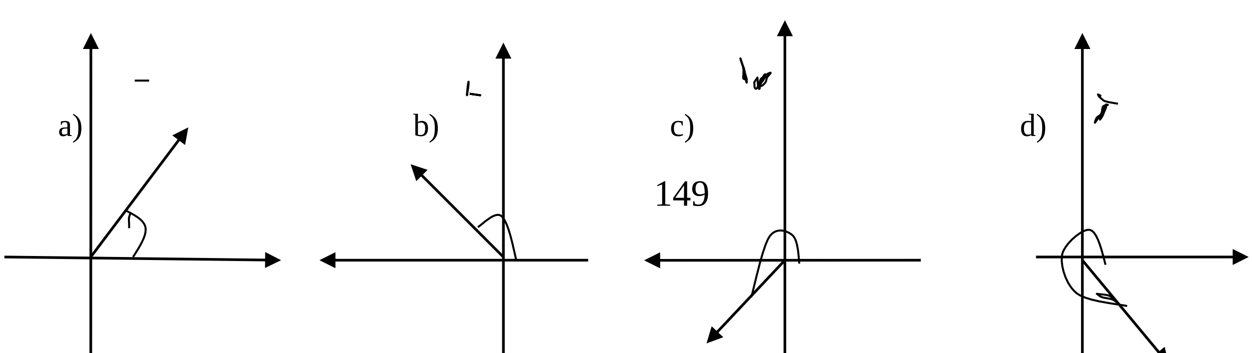
Angles in standard position

An angle in coordinate plane is said to be in standard position, if

- Its vertex is at the origin and
- Its initial side lies along the positive x-axis.

The terminal side of an angle in standard position lie either I, II, III or IV quadrant.

Example: - Considered the following angles in standard position where their terminal side is in the I, II, III or IV quadrants respectively.



1^{st}quadrant angle 2^{nd} *quadrant angle* 3^{rd} *quadrant angle* 4^{th} *quadrant angle*

Definition:- Angles in standard position having the same terminal side are called **co-terminal angles.**

i.e. let$\boldsymbol{\theta}$ is an angle in standard position then $\boldsymbol{\theta + n.360°}$, $\boldsymbol{n} \in \boldsymbol{Z}$, is co - terminal with θ

For example $45°$ *and* $-315°, 60°$ *and* $420°$, *are pair of coterminal angles.*

- If the terminal side of an angle in standard position lies in the first quadrant, then it is called the **first quadrant angle**.
- If the terminal side of an angle in standard position lies in the second quadrant, then it is called the **second quadrant angle**. Similar for third and fourth quadrant angles. The above figure indicates the example of first, second, third and fourth quadrant angle respectively.

Example: - In which quadrant do the following angles lie?

$$a)\ 420° b) 1272° c) -296° d) -4020°$$

Solution: - $a) 420° = 360° + 60°$ that means $420°$ is one complete revolution and an additional of $60°$. Hence, $420°$ *and* $60°$ *arecoterminalangles*. Since $60°$ lies in the first quadrant, $420°$ also lies in the first quadrant.

$d) -4020° = 5(-360°) + (-240°)$, This implies $-4020°$ *and* $-240°$ have the same terminal side in the second quadrant. Therefore $-4020°$ lies in the second quadrant.

b and c are lef for you.

Definition:-Angles in standard position whose terminal side coincide with the coordinate axes are known as **quadrant angles**.

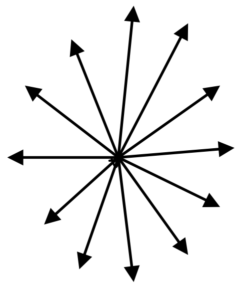
Example:$\ldots . -360°, -270°, -180°, -90°, 0°, 90°, 180°, 270°, 360°, \ldots \ldots . n(90°)$ *wheren* $\in \mathbb{Z}$ are quadrant angles.

The unit of measurements of angles

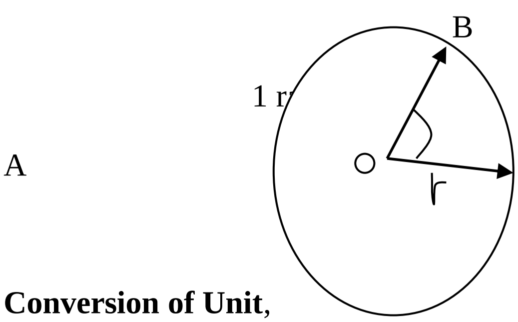
Angles are commonly measured by using two different units known as **degree** and **radian**.

Degree(^{0})**:-**consider that on a plane rays are drawn from the same point say O, in such a way that whole plane is divided by these rays in to 360^0 congruent angles and their interiors.

Inthe figure below, if $<$ AOBisoneoftheseangles , thenitsdegreemeasuretakento beonedegree(1^0).



Radian: -the measure of central angle of a circle subtended by an arc whose length is equal to the radius of circle. In the figure given below, if the length of minor arc AB is r the measure of the angle AOB is taken to be one radian, (1rad).



Conversion of Unit,

To convert one unit to other , we have the fact that the circumference of a circle with radius r is given by $2\pi r.$ which means the circumference is measured using the radius as a unit ,it is clear that circumference of a circle is 2π. this means again the circumference is$2\pi\ rdians.$ noting that the whole circle subtends $360^0 then$,

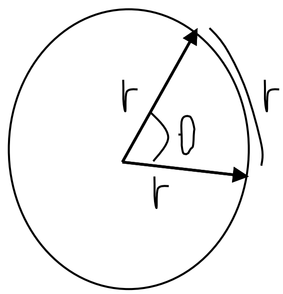
$$2\pi\ rad = 360^0 \Rightarrow \pi\ rad = 180^0.$$

$$\Rightarrow 1rad = \frac{180^0}{\pi} \cong 57.3^0$$

$$360^0 = 2\pi\, rad \Rightarrow 1^0 = \frac{2\pi}{360} rad$$

$$\Rightarrow 1^0 = 0.0175\, rad$$

Equivalently, the angle θatthecenterofacirclesubtendedbyanarc equalinlengthwith radius is**1radian**. Thatisθ $= \frac{r}{r} = 1$rad , asshowninthefigurebelow .



In general if the length of arc is *l* units and the radius is r unit, then θ $= \frac{l}{r}$radians

This indicates the size of the angle O is the ratio of the arc length to the radian of the circle.

Example: if $l = 5cm\ , r = 2cm calculate \theta in radian$

Solution

$$\theta = \frac{l}{r} = \frac{5}{2} = 2.5\text{rad}$$

Example: - convert the following degrees to radians, using the above relation

a) $360^0 b) 180^0$

Solution ;- a) Since$360^0 = 2\pi r, the length l of circumference of a circle with radius$

r is $2\pi r then \theta = \frac{l}{r} = \frac{2\pi r}{r} = 2\pi\ . That means 360^0 = 2\pi$

$b)$***Exercise*** !

In general, to convert degree to radian we multiply the given degree measure by $\frac{\pi}{180^0}$.

$$i.e \text{Radians} = \text{given degreeX}\frac{\pi}{180^0} \text{andDegree} = \text{ginen radiansX}\frac{180^0}{\pi}$$

Example: - convert each the following degree to radians

a) $60^0 b) - 150^0 c\ ) \ 240^0 \ \ d) - 180^0$

Solution:a) $60^0 = 60^0 \text{X} \frac{\pi}{180^0} = \frac{\pi}{3}\text{rad} \text{b)} - 150^0 = -150^0 \text{X} \frac{\pi}{180^0} = -\frac{5\pi}{6}\text{rad}$

$\boldsymbol{canddExercise}$!

Example ; converteachofthefollowingradianstodegree .

$$a) \ \frac{\pi}{12} b) \ \ \frac{-\pi}{6} c) \ \frac{2\pi}{3} d) \frac{-10\pi}{3}$$

Solution : a) $\frac{\pi}{12} = \frac{\pi}{12} X \frac{180^0}{\pi} = \frac{180^0}{12} = 15^0 d) \frac{-10\pi}{3} = \frac{-10\pi}{3} X \frac{180^0}{12} = -600^0$

b and c left for exercise!

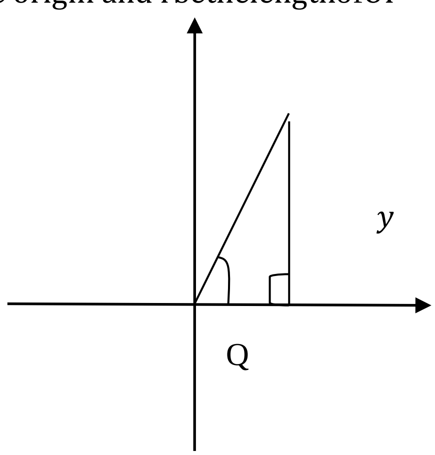
3.3.2.1Definition of Trigonometric Functions

Let θbeanyangleinstandardposition, p(x, y)beapointonthe terminal sideof the angle θ

different from the origin and rbethelengthofOP $= \sqrt{x^2 + y^2}$, asshownbelow .

yP (x, y)

θx



Based on the above fact the trigonometric functions are defined as follows:

$$\text{sine}\theta = \sin\theta = \frac{\text{oppositeside}}{\text{hypotenusside}} = \frac{\text{y}}{\text{r}}$$

$$\text{cosine}\theta = \cos\theta = \frac{\text{adjacentside}}{\text{hypotenuseside}} = \frac{x}{r}$$

$$\text{tangent}\theta = \tan\theta = \frac{\text{oppositeside}}{\text{adjacentside}} = \frac{y}{x}$$

The other three functions, cosecant, secant and cotangent are related reciprocally with the sine, cosine and tangent functions respectively.

$$\text{cosecant}\theta = \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

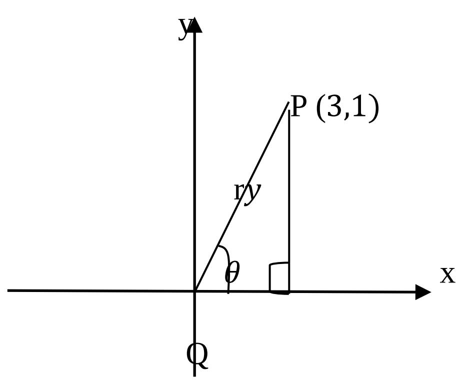
$$\text{secant}\theta = \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\text{cotangent}\theta = \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

Example: suppose point$p(3,2)$ is on the terminal side of angle θ, thenfindthevaluesofthe

sixtrigonometricfunctions.

Solution: - consider the figure given below,



$$p(x, y) = (3,1)$$

$$\Rightarrow x = 3\ and y = 1$$

Use Pythagoras' theorem to find r;

$$x^2 + y^2 = r^2 3^2 + 1^2 = r^2$$

$$r = \sqrt{10}units$$

Then $\sin\theta = \frac{\text{oppositesideof}\theta}{\text{hypotenusesideof}\theta} = \frac{2}{r} = \frac{2}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{10}}{5}} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}$

$\cos\theta = \frac{\text{adjacentsideofangle}\theta}{\text{hypotenusesideofangle}\theta} = \frac{x}{r} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3\sqrt{10}}{10}} = \frac{10}{3\sqrt{10}} = \frac{\sqrt{10}}{3}$

$$\tan\theta = \frac{y}{x} = \frac{2}{3}, \cot\theta = \frac{1}{\tan\theta} = \frac{3}{2}$$

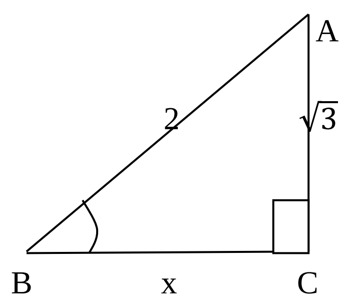
Note; any co – terminal angles have the same trigonometric values

- $\mathbf{\sin\theta = \sin(\theta + n.360°)}$ **and** $\mathbf{\csc\theta = \csc(\theta + n.360°)}$
- $\mathbf{\cos\theta = \cos(\theta + n.360°)}$ **and** $\mathbf{\sec\theta = \sec(\theta + n.360°)}$
- $\mathbf{\tan\theta = \tan(\theta + n.360°)}$ **and** $\mathbf{\cot\theta = \cot(\theta + n.360°)}$

Where, n is an element of natural number**.**

Example; $-Given$ $\triangle ABC$ right angled at C. AB=2, AC=$\sqrt{3}$, thenfindthebasic

trigonometricvaluesforangleB.



Solution: first you should find the value of BC, given that AB=2, AC=$\sqrt{3}$ and BC= x

Now, by Pythagoras theorem, we have

$$(AC)^2 + (BC)^2 = (AB)^2, butAC = \sqrt{3}andAB = 2$$

$$\Rightarrow x^2 + \left(\sqrt{3}\right)^2 = (2)^2, but\sqrt{x^2} = xforx \geq 0$$

$$\Rightarrow x^2 + 3 = 4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

But the length cannot be negative; we have to take only positive value of x

$$\Rightarrow x = 1$$

Therefore , $Ac = \sqrt{3}$, $BC = 1$ and $AB = 2$

Hence, $\cos B = \frac{BC}{AB} = \frac{1}{2} \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{BC}{AB}} = \frac{AB}{BC} = 2$

$$\sin B = \frac{AC}{AB} = \frac{\sqrt{3}}{2} \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{AC}{AB}} = \frac{AB}{AC} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan B = \frac{AC}{BC} = \frac{\sqrt{3}}{1} = \sqrt{3}\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{AC}{BC}} = \frac{BC}{AC} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example: Let $Sin30^0 = \frac{1}{2}$, $cos30^0 = \frac{\sqrt{3}}{2}$, then find the remaining trigonometric .Values of 30^0

i. $\tan\theta = \sin\theta/\cos\theta \Rightarrow \tan\theta = \frac{\sin 30^0}{\cos 30^0} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

ii. $\csc\theta = \frac{1}{\sin\theta} \Rightarrow \csc 30^0 = \frac{1}{\sin 30^0} = \frac{1}{\frac{1}{2}} = 2$

iii. $\sec\theta = \frac{1}{\cos\theta} \Rightarrow \sec 30^0 = \frac{1}{\cos 30^0} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}}{3}$

iV. $\cot\theta = \frac{1}{\tan\theta} \Rightarrow \cot 30^0 = \frac{1}{\tan 30^0} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$or

$$. \cot\theta = \frac{1}{\tan\theta} \frac{1}{\tan 30^0} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

$$\cot 30^0 = \frac{\cos 30^0}{\sin 30^0} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} x2 = \sqrt{3}$$

Signs of trigonometric functions

	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
Sin	+	+	-	-
Cos	+	-	-	+
Tan	+	-	+	-
Csc	+	+	-	-
Sec	+	-	-	+
Cot	+	-	+	-

3.4.2 Trigonometric values of Angles

Trigonometric values of Angles, some special angles($(0^0 \leq \theta \leq 360^0)$

θ	radian	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0^0	0	0	1	0	undefined	1	undefined
30^0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{2}$	$\sqrt{3}$
45^0	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60^0	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90^0	$\frac{\pi}{2}$	1	0	undefined	1	Undefined	0
120^0	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{-1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$\frac{-\sqrt{3}}{3}$
135^0	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150^0	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{3}$	2	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{3}$
180^0	π	0	-1	0	undefined	-1	undefined
210^0	$\frac{7\pi}{6}$	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$\frac{-2\sqrt{3}}{3}$	$\sqrt{3}$
225^0	$\frac{5\pi}{6}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	**1**
240^0	$\frac{4\pi}{3}$	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	$\sqrt{3}$	$\frac{-2\sqrt{3}}{3}$	-2	$\frac{\sqrt{3}}{3}$
270^0	$\frac{3\pi}{2}$	-1	0	undefined	-1	Undefined	0
300^0	$\frac{5\pi}{3}$	$\frac{-\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$\frac{-2\sqrt{3}}{3}$	2	$\frac{-\sqrt{3}}{3}$
315^0	$\frac{7\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330^0	$\frac{11\pi}{6}$	$\frac{-1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{2}$	-2	$\frac{-\sqrt{3}}{3}$	$-\sqrt{3}$
360^0	2π	0	1	0	undefined	1	undefined

3.4 .3. Relationships between Trigonometric Functions

There are different relations between the trigonometric functions. Some of the relations are reciprocal relations, co function relations, ratio relation and Pythagoras relations. Each of the relations is discussed as follows:

Reciprocal Relation

- Sine and cosecant functions have reciprocal relation.

$$\sin\Theta = \frac{y}{r} \text{ and its reciprocal is } \csc\Theta = \frac{r}{y}$$

Hence, $\sin\Theta = \frac{1}{\csc\Theta}$ or $\csc\Theta = \frac{1}{\sin\Theta}$

- Cosine and secant functions have reciprocal relations.

$$cos\theta = \frac{x}{r} \; and\ its\ reciprocal\ is\ sec\theta = \frac{r}{x}$$

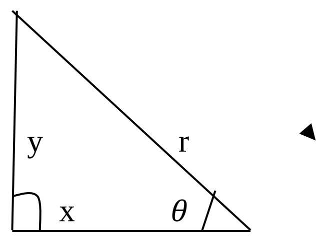
Hence, $\cos\Theta = \frac{1}{\sec\Theta}$ or $\sec\Theta = \frac{1}{\cos\Theta}$

- Tangent and cotangent functions have reciprocal relations.

$$tan\theta = \frac{y}{x} \; and\ its\ reciprocal\ is\ cot\theta = \frac{x}{y}$$

Hence, $\tan\Theta = \frac{1}{\cot\Theta}$ or $\cot\Theta = \frac{1}{\tan\Theta}$

The Co function Relation



Consider the above figure;

$$sin\theta = \frac{y}{r} = cos\alpha, \; tan\theta = \frac{y}{x} = cot\alpha \text{ and } csc\theta = \frac{r}{y} = \sec\alpha$$

The above relation of six trigonometric functions is known as **co-function relations**.

The co-functions of complementary angles are equal.

Suppose $\theta + \alpha = 90° \; or \; \theta = 90° - \alpha$

$$\sin\Theta = \cos\alpha \text{ or } \sin\alpha = \cos\Theta \text{ equivalently } \sin\Theta = \cos(90° - \Theta)$$

$$\tan\Theta = \cot\alpha \text{ or } \tan\alpha = \cot\Theta \text{ equivalently } \tan\Theta = \cot(90° - \Theta)$$

$$\sec\Theta = \csc\alpha \text{ or } \csc\alpha = \sec\Theta \text{ equivalently } \sec\Theta = \csc(90° - \Theta)$$

Example: sin60° = cos30° or sin30° = cos60°

$\tan 60° = \cot 30°$ or $\tan 30° = \cot 60°$

$\sec 60° = \csc 30°$ or $\sec 30° = \csc 60°$

Activity1.16

a. Let $\sin 52° = \cos\Theta$, then $\Theta = -----$

b. Let $\tan\Theta = \cot\propto$ if $\propto = 35°$, then $\Theta = ----$

c. Let $\csc 75° = \sec y$, then $y = -----$

The Ratio Relations

The ratio of two trigonometric functions yields another trigonometric function.

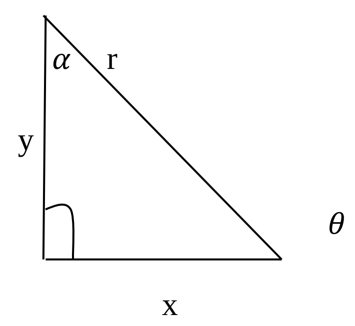
That is; $\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\Theta$

$\frac{\cos\theta}{\sin\theta} = \frac{x/r}{y/r} = \frac{x}{y} = \cot\Theta$

The Pythagoras Relation

From the Pythagoras Theorem on right angled triangle we have

$x^2+y^2 = r^2$



Dividing each term of the above equation by r^2

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}=\frac{r^2}{r^2} \Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

Since, $\cos\Theta = \frac{x}{r}$ and $\sin\Theta = \frac{y}{r} \Rightarrow$ $\mathbf{\cos^2\Theta + \sin^2\Theta = 1}$

Dividing the above equation by x^2

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2} \Rightarrow 1+ \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

Since, $\tan\Theta = \frac{y}{x}$ and $\sec\Theta = \frac{r}{x} \Rightarrow \mathbf{1+\tan^2\Theta = \sec^2\Theta}$

Dividing by y^2

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2} \Rightarrow \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

Since, $\cot\Theta = \frac{x}{y}$ and $\csc\Theta = \frac{r}{y} \Rightarrow \mathbf{\cot^2\Theta+1 = \csc^2\Theta}$

Example: If the point P(x, y) has coordinates (4,3), then find the ratio of six trigonometric values.

Solution:

The point P (4,3) has both coordinates positive implies p is a point in the first quadrant. Hence, the values of all the trigonometric ratios are positive.

Now, $r^2 = x^2+y^2 \Rightarrow 4^2+3^2=r^2 \Rightarrow r^2 = 16+9 =25 = \Rightarrow r =|\pm 5| \Rightarrow r = 5$

$\cos\Theta = \frac{x}{r} = \frac{4}{5} \Rightarrow \sec\Theta = \frac{r}{x} = \frac{5}{4}$

$$\sin\Theta = \frac{y}{r} = \frac{3}{5} \Rightarrow \csc\Theta = \frac{r}{y} = \frac{5}{3}$$

$$\tan\Theta = \frac{y}{x} = \frac{3}{4} \Rightarrow \cot\Theta = \frac{x}{y} = \frac{4}{3}$$

Example: Given $\cos\theta = \frac{-3}{5}$ for Θ second quadrant angle, find the values of remaining basic trigonometric ratios.

Solution: from Pythagoras relation we have

$$\text{Cos}^2\Theta+\sin^2\Theta =1 \quad \Rightarrow \sin^2\Theta = 1-\cos^2\Theta$$

$$\Rightarrow \sin^2\Theta = 1-\left(\frac{-3}{5}\right)^2 = 1-\frac{9}{25} \Rightarrow \sin^2\Theta = \frac{16}{25}$$

$$\Rightarrow \sin\Theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

But Θ is second quadrant angle so sinΘ is positive

$$\text{Hence, } \sin\Theta = \frac{4}{5}$$

Now, $\sec\Theta = \frac{1}{cos\theta} = \frac{1}{-3/5} = \frac{-5}{3}$

$\csc\Theta = \frac{1}{sin\theta} = \frac{1}{4/5} = \frac{5}{4}$

$\tan\Theta = \frac{sin\theta}{cos\theta} = \frac{4/5}{-3/5} = \frac{-4}{3}$

$\cot\Theta = \frac{\mathbf{1}}{\boldsymbol{tan\theta}} = \frac{\mathbf{1}}{\mathbf{-4/3}} = \frac{\mathbf{-3}}{\mathbf{4}}$

3.4.4. The Sum Angle Formula

Theorem: for any two angles θ and β

- $\cos(\theta \pm \beta) = \cos\theta\cos\beta \mp \sin\theta\sin\beta$
- $\sin(\theta \pm \beta) = \sin\theta\cos\beta \pm \cos\beta\sin\theta$
- $\tan(\theta \pm \beta) = \frac{\tan\theta \pm \tan\beta}{1 \mp \tan\theta\tan\beta}$ whenever $\sin\theta \neq 0, \sin\beta \neq 0, \cos\theta \neq 0$ and $\cos\beta \neq 0$

Example; express $\cos 30^0$ as the sum and difference angle formula

Solution;

- $\cos 30^0 = \cos(20^0 + 10^0) = \cos 20^0 \cos 10^0 - \sin 20^0 \sin 10^0$
- $\cos 30^0 = \cos(15^0 + 15^0) = \cos 15^0 \cos 15^0 - \sin 15^0 \sin 15^0$

$$= (\cos 15^0)^2 - (\sin 15^0)^2$$

- $\cos 30^0 = \cos(45^0 - 15^0) = \cos 45^0 \cos 15^0 + \sin 45^0 \sin 15^0$
- $\cos 30^0 = \cos(60^0 - 30^0) = \cos 60^0 \cos 30^0 + \sin 60^0 \sin 30^0$

Example; Expresssin 20^0 as the sum and difference angle formula

Solution;

- $\sin 20^0 = \sin\ (10^0 + 10^0) = \sin 10^0 \cos 10^0 + \cos 10^0 \sin 10^0 = 2 \sin 10^0 \cos 10^0$
- $\sin 20^0 = \sin\ (15^0 + 5^0) = \sin 15^0 \cos 5^0 + \cos 15^0 \sin 5^0$
- $\sin 20^0 = \sin\ (45^0 - 25^0) = \sin 45^0 \cos 25^0 - \cos 45^0 \sin 25^0$
- $\sin 20^0 = \sin\ (60^0 - 40^0) = \sin 60^0 \cos 40^0 - \cos 60^0 \sin 40^0$

Example; Express$\sin 20^0$ as the sum and difference angle formula

Solution;

- $\tan 10^0 = \tan(9^0 + 1^0) = \frac{\tan 9^0 + \tan 1^0}{1 - \tan 9^0 \tan 1^0}$
- $\tan 10^0 = \tan(7^0 + 3^0) = \frac{\tan 7^0 + \tan 3^0}{1 - \tan 7^0 \tan 3^0}$
- $\tan 10^0 = \tan(13^0 - 3^0) = \frac{\tan 13^0 - \tan 3^0}{1 + \tan 13^0 \tan 3^0}$

Activity 3.17

Express each of the following as a single trigonometric value.

a. $\sin 40^0 \cos 20^0 + \cos 40^0 \sin 20^0$
b. $\sin 40^0 \cos 20^0 - \cos 40^0 \sin 20^0$
c. $\sin A \cos 2A + \cos A \sin 2A$
d. $\cos 50^0 \cos 20^0 + \sin 50^0 \sin 20^0$
e. $\cos A \cos 2A - \cos A \cos 2A$
f. $\frac{\tan 57^0 - \tan 12^0}{1 + \tan 57^0 \tan 12^0}$

Note;

- $\cos(-\theta) = \cos\theta$ and its domain is symmetric;
  cosine function is an even function
- $\sin(-\theta) = -\sin\theta$ and its domain is symmetric;
  Sine function is an odd function
- $\tan(-\theta) = -\tan\theta$ and its domain is symmetric;
  Tangent function is an odd function

Example; Find the exact value of;

a. $\sin 75^0$ b. $\tan 15^0 + \cot 15^0$

Solution; $75^0 = 45^0 + 30^0$(as a sum two special angles)

Hence,

a. $\sin 75^0 = \sin(45^0 + 30^0) = \sin 45^0 \cos 30^0 + \cos 45^0 \sin 30^0$

$$= \frac{\sqrt{2}}{2} \text{x} \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \text{x} \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$15^0 = 45^0 - 30^0$$

b. $\tan 15^0 = \tan(45^0 - 30^0) = \frac{\tan 45^0 - \tan 30^0}{1 + \tan 45^0 \tan 30^0} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\text{x}\frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$$\cot 15^0 = \frac{1}{\tan 15^0} = \frac{1}{\frac{\sqrt{3}-1}{\sqrt{3}+1}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Hence;

$$\tan 15^0 + \cot 15^0 = \frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \text{x} \frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} \text{x} \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= 2 - \sqrt{3} + 2 + \sqrt{3} = 4$$

Double Angle and Half Angle Formula

Double Angle formula

Theorem; for any angle $\theta \in \mathbb{R}$

- $\cos 2\theta = \begin{cases} 2cos^2\theta - 1 \\ cos^2\theta - sin^2\theta \\ 1 - 2sin^2\theta \end{cases}$
- $\sin 2\theta = 2 \sin\theta \cos\theta$
- $\tan 2\theta = \frac{2\tan\theta}{1 - tan^2\theta}$

Example; Let us describe each of the following angles, using the double angle formula;

a. $\cos 4A$ *and* $\cos 70^0$

b. $\sin 4\text{A}$ *and* $\sin 120^0$

c. $\tan 4A$ *and* $\tan 40^0$

Solution;

$$\cos 4\text{A} = \begin{cases} 2cos^2\frac{1}{2}(4A) - 1 = 2cos^2 2A - 1 \\ cos^2\frac{1}{2}(4A) - sin^2\frac{1}{2}(4A) = cos^2 2A - sin^2 2A \\ 1 - 2sin^2\frac{1}{2}(4A) = 1 - 2sin^2 2A \end{cases}$$

$\cos 70^0$, **Exercise**

$$\sin 4\text{A} = 2\sin\frac{1}{2}(4A)\cos\frac{1}{2}(4A) = 2\sin 2A\cos 2A$$

$\sin 120^0$; **Exercise**

$$\tan 40 = \frac{2\tan\frac{1}{2}(40)}{1 - tan^2\frac{1}{2}(40)} = \frac{2\tan 20}{1 - tan^2 20}$$

$\tan 4\text{A}$; **Exercise**

The half Angle Formula

Theorem; for any angle $\theta \in \mathbb{R}$

$$\sin\frac{\theta}{2} = \begin{cases} \sqrt{\frac{1-\cos\theta}{2}}; \ \ for\ 0 < \frac{\theta}{2} < 180^0 \\ -\sqrt{\frac{1-\cos\theta}{2}}; for\ 180^0 < \frac{\theta}{2} < 180^0 \end{cases}$$

$$\cos\frac{\theta}{2} = \begin{cases} \sqrt{\frac{1+\cos\theta}{2}}; \ \ for\ \frac{\theta}{2}\ in\ 1^{st} and 4^{th} quadrants \\ -\sqrt{\frac{1+\cos\theta}{2}}; for\ \frac{\theta}{2} 2^{nd} and\ 3^{rd} quadrants \end{cases}$$

$$\tan\frac{\theta}{2} = \begin{cases} \dfrac{1-\cos\theta}{sin\theta}; \ \sin\theta \neq 0, (\theta \neq 0^0, 180^0) \\ \dfrac{\sin\theta}{1+\cos\theta}, \cos\theta \neq -1, (\theta \neq 180^0) \end{cases}$$

Example; Let us describe each of the following angles, using the double angle formula

a. $\sin 6.5^0$ c. $\cos 22.5^0$ d. $\tan 22.5^0$

Solution;

a. $\sin 6.5^0 = \sin(\frac{13}{2})^0 = \sqrt{\frac{1-\cos 2(\frac{13}{2})}{2}} = \sqrt{\frac{1-\cos 13}{2}}$

b. $\cos 22.5^0 = \cos(\frac{45}{2})^0 \sqrt{\frac{1+\cos 2((\frac{45}{2})^0)}{2}} = \sqrt{\frac{1+\cos 45^0}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

c. $\tan 22.5^0 = \tan(\frac{45}{2})^0 = \frac{1-\cos 45}{sin45} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} \text{x} \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1$

3.4.5. Graph of Basic Trigonometric Function

Trigonometric functions are real valued functions. The independent variable is the angle x measured either in degree or radians and expressed in real numbers. As we already know graphs of functions of one variable are drawn using a coordinate system on a plane. Accordingly the horizontal or the x-axis of our coordinate system is scaled in radian units. The value of each function at a certain angle will correspond to the y – coordinate of a point on the graph of the function.

i. The Graph of the Sine Function

The sine function $y = sinx$ is defined for any real number x, so the domain of sine function is the set of real number and the range is the set of all real number between -1 and 1 inclusively and its graph iscontinuous and smooth curve. $\text{Sin}(-x) = -\text{Sinx}$ and its domain $\mathbb{R}$ is symmetric this indicates that sine function is an***odd function*** and the graph issymmetric about the ***origin***. Moreover $\text{sinx} = \sin(x + 2n\pi)$ for $n \in \mathbb{R}$, this implies sine function is periodic function with ***period*** $\boldsymbol{2\pi}$**.**

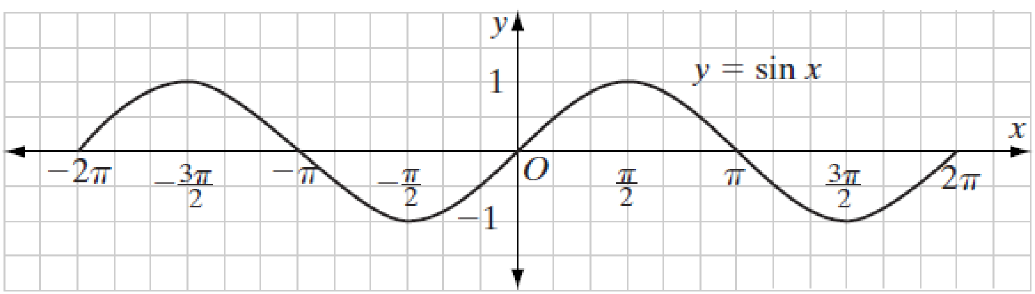
To draw the graph of sine function, we investigate how $sinx$ behaves as the independent variable x increase from -2π to 2π

- As x increases from -2 π to $\frac{-3\pi}{2}$ and 0 to $\frac{\pi}{2}$, $sinx$ increases from 0 to 1.
- As x increases from $\frac{-3\pi}{2}$ to $-\pi$ and $\frac{\pi}{2}$ to π, $sinx$ decreases from 1 to 0.
- As x increases from $-\pi$ to $\frac{-\pi}{2}$ and π to $\frac{3\pi}{2}$, $sinx$ decreases from 0 to -1.
- As x increases from $\frac{-\pi}{2}$ to 0 and $\frac{3\pi}{2}$ to 2π, $sinx$ increases from -1 to 0.

Now let as summarize the above facts by the table as follows,

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
sinx	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sinx	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

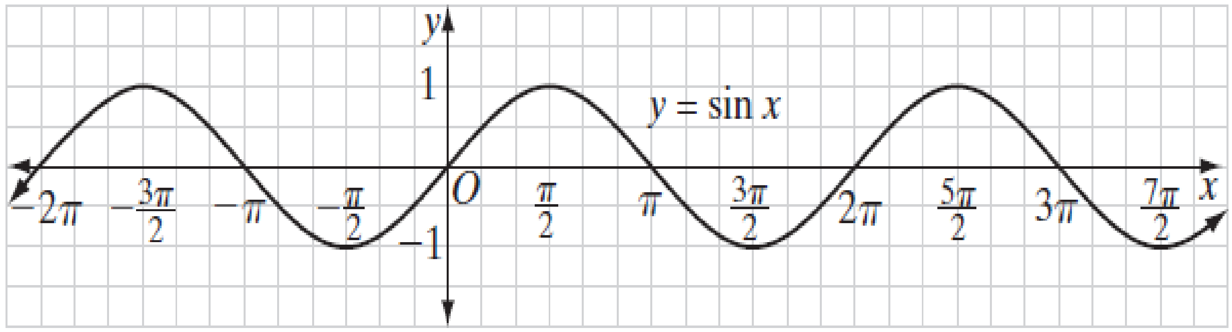
Now to sketch the graph of sine function, we will plot the coordinate points on the coordinate plane and joined by a smooth curve as follows.



The graph of $y = sinx$ on the interval $-2\pi \leq x \leq 2\pi$

In each time we increase or decrease the value of the x-coordinates by a multiple of 2π, the sine graph is reapeated and each portion of the graph in an interval of 2π is ***one cycle*** of the sine function.

In general the graph of a sine function throughout its domain has the following forms,



The Graph of $y = sinx$

ii. The Graph of the Cosine Function

The cosine function $y = cosx$ is defined for any real number x, so the domain of cosine function is the set of real number and the range is the set of all real number between -1 and 1 inclusively and its graph iscontinuous and smooth curve. $\cos x = \cos(-x)$and its domain $\mathbb{R}$ is symmetric this indicates that cosine function is an***even function*** and the graph is symmetric about the ***y-axis***. Moreover $\cos x = \cos(x + 2n\pi)$ for $n \in \mathbb{R}$, this implies cosine function is periodic function with ***period*** $\boldsymbol{2\pi}$.

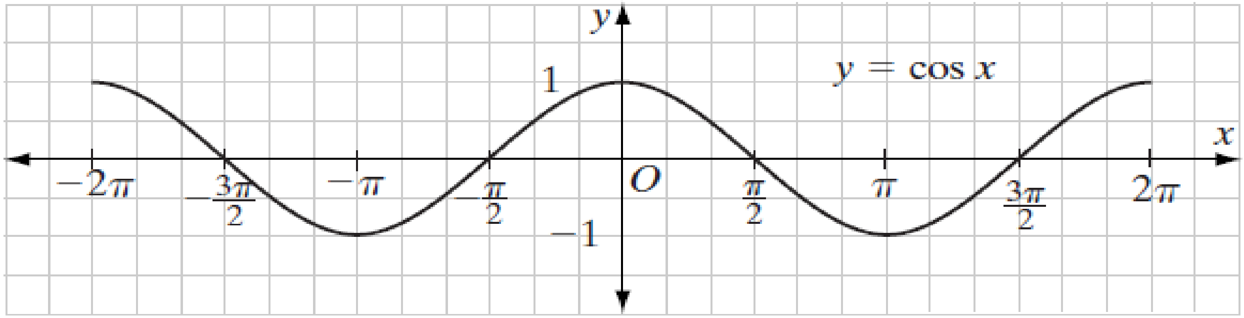
To draw the graph of cosine function, we investigate how $cosx$ behaves as the independent variable x increase from -2π to 2π

- As x increases from -2 π to$-\pi$ and 0 to π, $cosx$ decreases from 1 to -1.
- As x increases from $-\pi$ to 0 $and\ \pi$ to2π, $cosx$ increases from -1 to 1.

Now let as summarize the above facts by the table as follows,

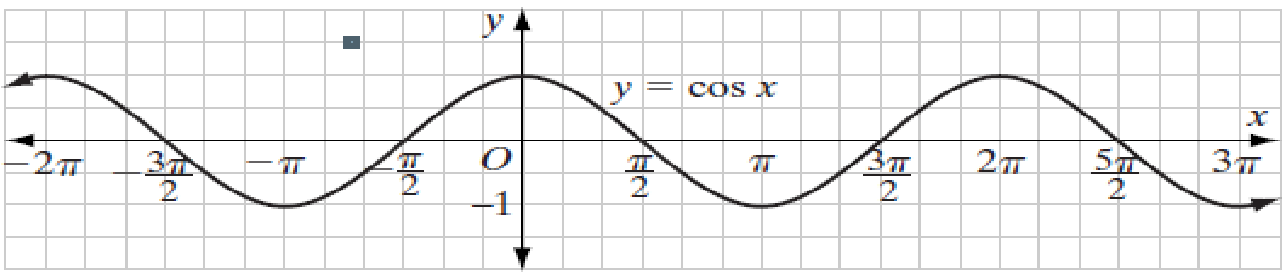
x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
cosx	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cosx	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1

Now to sketch the graph of cosine function, we will plot the coordinate points on the coordinate plane and joined by a smooth curve as follows.



The Graph of $y = cosx$ on the interval $-2\pi \leq x \leq 2\pi$

In each time we increase or decrease the value of the x-coordinates by a multiple of 2π, the cosine graph is reapeated and each portion of the graph in an interval of 2π is ***one cycle*** of the cosine function. In general the graph of a cosine function throughout its domain has the following forms,



The Graph of $y = cosx$

iii. The graph of Tangent Function

The tangent function $y = tanx$ is defined for all x except the value of x that make$cosx = 0$ and its range is any real number. Tangent function is always an increasing function throughout its domain and it is periodic function with periodπ. The tangent function is an odd function; since it satisfies the condition $\tan(-x) = -\tan x$ and its graph has symmetric about the origin with vertical asymptotes at

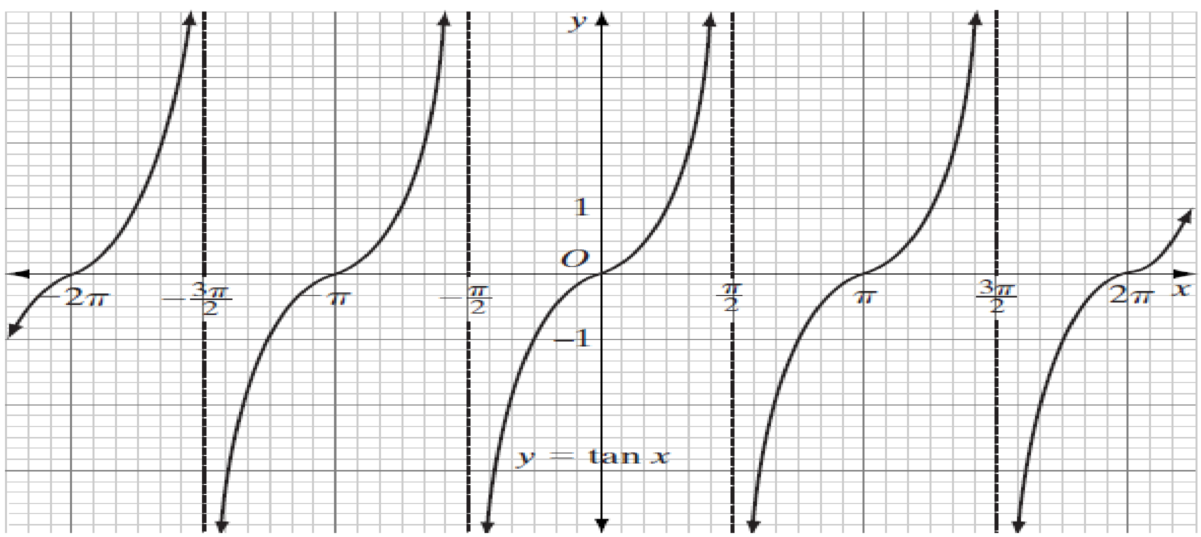
$$x = \frac{n\pi}{2}$$

Where n an element of odd integer

That is the graph of tangent function is a curve that increases through negative value of $\tan x$ to 0 and then continues to increase through positive value. At odd multiple of $\frac{\pi}{2}$ the graph is discontinues and then repeats the same pattern, since there is one complete cycle of the curve in the interval $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ tangent function is periodic with period π

Now we can use the table below to draw the graph of y = tanx

x	-2π	$\frac{-11\pi}{6}$	$\frac{-5\pi}{3}$	$\frac{-3\pi}{2}$	$\frac{-4\pi}{3}$	$\frac{-7\pi}{6}$	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0
tanx	0	0.58	1.73	und	-1.73	-0.58	0	0.58	1.73	und	-1.73	-0.58	0
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
tanx	0	0.58	1.73	und	-1.73	-0.58	0	-0.87	0.58	und	-1.73	-0.58	0



The graph of $y = tanx$

iv. The Graph of Cosecant Function

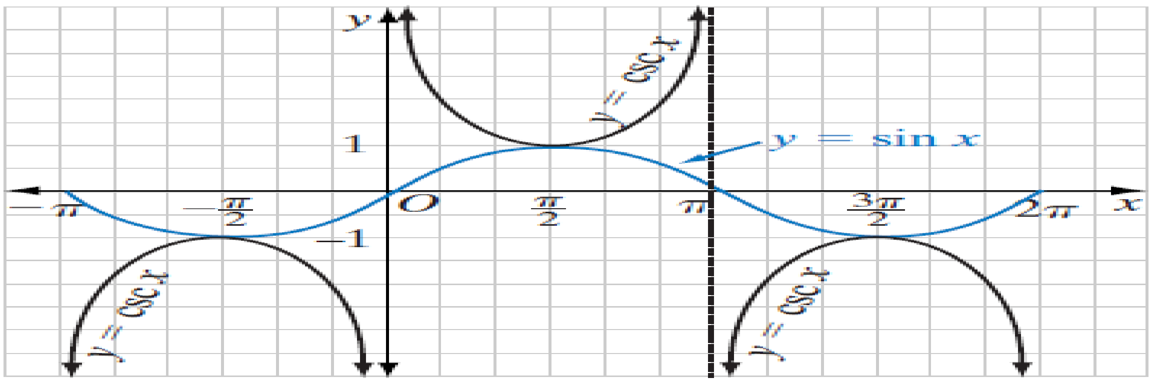
The cosecant function is defined in terms of sine function;

$$\csc x = \frac{1}{\sin x}$$

Since $-1 \leq \sin x \leq 1$, therefore $-\infty < \csc x \leq -1$ and $1 \leq \csc x < \infty$

That is; the domain of cosecant function is all values of x except the value of x that makes $\sin x = 0$ and the range of cosecant function is $-\infty < \csc x \leq -1$ and $1 \leq \csc x < \infty$

To draw the graph of the cosecant function, we can use the reciprocal of the sine function values. The reciprocal of 1 is 1; the reciprocal of a positive number less than one is greater than 1; the reciprocal of -1 is -1 and the reciprocal of a negative number greater than -1 is less than -1. If $x = n\pi$ where$n \in \mathbb{Z}$, $\sin x = 0$ and $\csc x$ is undefined, then the vertical line $x = n\pi$ are vertical asymptotes for the graph of cosecant function. The graph of cosecant function on the interval $-\pi$ to 2π is as follows



The Graph of $y = \csc x$ on the interval $-\pi$ to 2π

Activity 3.18

Draw the graph of

a. $y = \sec x$ b. $y = \cot x$

3.4.6. The Sine and Cosine Laws

Any triangles are seven elements; these are the three sides, the three angles and one area. Solving a triangle means finding the missing part of these seven elements. If the triangle is right angle triangle; it is easy to solve by using Pythagoras Theorem, but it is difficult to solve acute and obtuse angle triangle. To reduce this problem let us drive sine law, cosine law and area law.

The Law of Sine

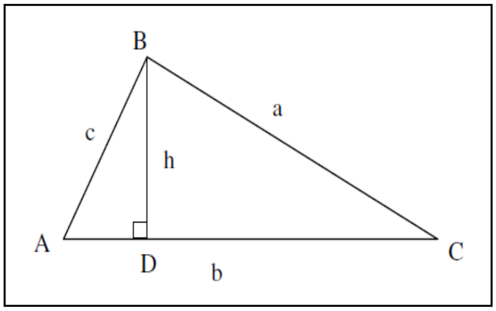
If A, B, and C are the measures of the angles of any triangle and if a, b, and c are the lengths of the sides opposite the corresponding angles, then

$$\frac{\mathrm{c}}{\sin \mathrm{C}} = \frac{\mathrm{a}}{\sin \mathrm{A}} = \frac{\mathrm{b}}{\sin \mathrm{B}} \quad or \quad \frac{\sin \mathrm{C}}{\mathrm{c}} = \frac{\sin \mathrm{A}}{\mathrm{a}} = \frac{\sin \mathrm{B}}{\mathrm{b}}$$

Proof; Triangle **ABC** is a non-right angled triangle.

In triangle **ABC**,

Draw a perpendicular line from B to AC meeting AC at D. This creates two right angled triangles **ABD** and **BDC**



In triangle **ABD**:

$$\sin \mathrm{A} = \frac{h}{c} \implies h = c \sin \mathrm{A} ------------------ (*)$$

In triangle **BDC:**

$$\sin \mathrm{C} = \frac{h}{a} \implies h = a \sin \mathrm{C} ----------------- (**)$$

From $(*)$ and $(**)$

$$a \sin \mathrm{C} = c \sin \mathrm{A}$$

We get

$$\frac{a}{\sin \mathrm{A}} = \frac{c}{\sin \mathrm{C}} ---------------------- (1)$$

Similarly; draw a perpendicular line from C to AB meeting AB at E. This creates two right angled triangles **ACE** and **BEC. CE** $=$ **h′**is the new altitudefrom **C to AB.**

In triangle **ACE:**

$$\sin \mathrm{A} = \frac{\mathrm{h}'}{\mathrm{b}} \implies \mathrm{h}' = \mathrm{b} \sin \mathrm{A} ----------------- (\mathrm{i})$$

In triangle **BEC:**

$$\sin \mathrm{B} = \frac{\mathrm{h}'}{\mathrm{a}} \implies \mathrm{h}' = \mathrm{a} \sin \mathrm{B} ----------------- (\mathrm{ii})$$

From (i)and (ii)

$$a \sin \mathrm{B} = b \sin \mathrm{A}$$

We get;

$$\frac{a}{\sin \text{A}} = \frac{b}{\sin B} ------------------------(2)$$

From$(1) and$ (2); we get

$$\frac{\text{c}}{\sin \text{C}} = \frac{a}{\sin \text{A}} = \frac{b}{\sin B}$$

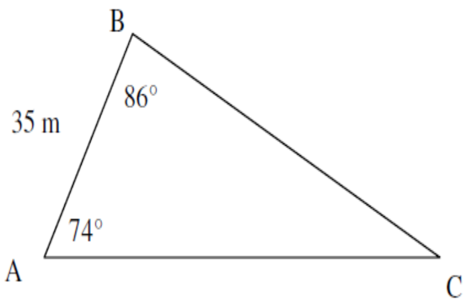
To use the sine rule, choose an appropriate pair, depending on what you know in the triangle.

$$\frac{\text{c}}{\sin \text{C}} = \frac{a}{\sin \text{A}} \text{or} \frac{a}{\sin \text{A}} = \frac{b}{\sin B} \text{ or } \frac{\text{c}}{\sin \text{C}} = \frac{b}{\sin B}$$

Due to the fact that the Law of Sine uses proportions that involve both angles and sides, the following pieces of information are needed in order to solve an oblique triangle using the Law of Sine:

- If two sides and the angle not included between them are given
- If two angles and the side between them are given.
- If two angles and one side that is not included in the angles.

Example; Find the length of BC in triangle ABC



Solution;

$$m(< A) + m(< B) + m(< C) = 180^0$$

$$74^0 + 86^0 + \ m(< C) = 180^0$$

$$160^0 + m(< C) = 180^0$$

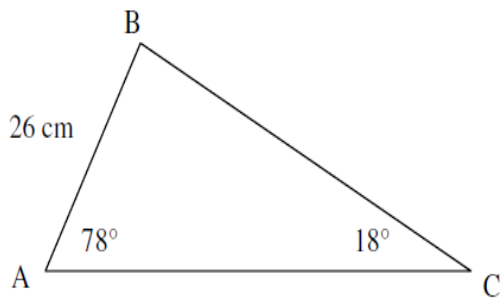
$$m(< C) = 180^0 - 160^0 = 20^0$$

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{35}{\sin 20^0} = \frac{BC}{\sin 74^0}$$

$$BC = \frac{35 \sin 74^0}{\sin 20^0} = \frac{35(0.9613)}{0.3420} = \frac{33.6455}{0.3420} \approx 98$$

Activity 3.19

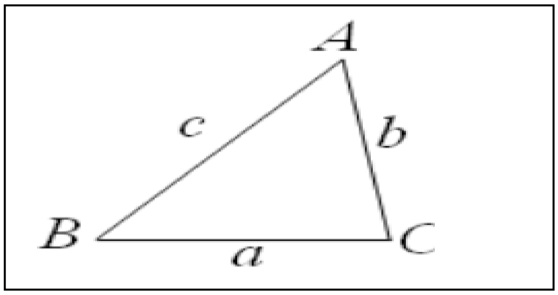
Find the length of AC in triangle ABC



The Law of Sine cannot be used directly to solve triangles if we know two sides and the angle between them or if we know all three sides. In these two cases, the **Law of Cosines** applies.

The Law of Cosine

In any triangle ABC, we have

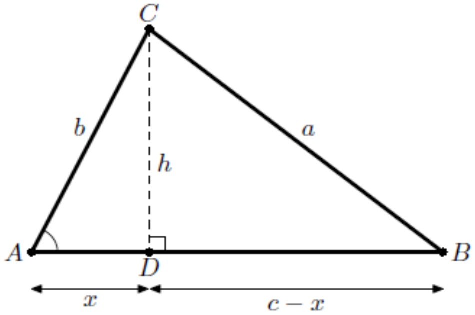


$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof; consider triangle ABC



From the right triangle ADC; we deduce

$$x^2 + h^2 = b^2 ----------------------(1)$$

And

$$\cos A = \frac{x}{b} \Longrightarrow x = b\cos A ---------------- (2)$$

From the right triangle BDC; we deduce

$$(c-x)^2 + h^2 = a^2 \Longrightarrow a^2 = c^2 - 2cx + (x^2 + h^2) ------(3)$$

Substituting equations $(1) and$ (2) in to (3), we get;

$$\mathrm{a}^2 = \mathrm{c}^2 - 2\mathrm{c}(\mathrm{b}\cos \mathrm{A}) + \mathrm{b}^2$$

$$\mathrm{a}^2 = \mathrm{b}^2 + \mathrm{c}^2 - 2\mathrm{bc}\cos \mathrm{A}$$

Similarly

$$\mathrm{b}^2 = \mathrm{a}^2 + \mathrm{c}^2 - 2\mathrm{ac}\cos \mathrm{B}$$

$$\mathrm{c}^2 = \mathrm{a}^2 + \mathrm{b}^2 - 2\mathrm{ab}\cos \mathrm{C}$$

Note; Use the law of cosine to solve triangles;

- If we know two sides and the angle between them
- If we know all three sides

Example; the length of sides of a triangle ABC are $a = 5, b = 8$ and $c = 12$. Find the measure of angles of a triangle.

Solution; $\mathrm{a}^2 = \mathrm{b}^2 + \mathrm{c}^2 - 2\mathrm{bc}\cos \mathrm{A} \quad \Longrightarrow \cos \mathrm{A} = \frac{\mathrm{b}^2+\mathrm{c}^2-\mathrm{a}^2}{2\mathrm{bc}} = \frac{8^2+12^2-5^2}{2\mathrm{x}8\mathrm{x}12} = \frac{183}{192} = 0.9531$

$$\mathrm{m}(< A) = \cos^{-1} 0.9531 \approx 18^0$$

$$\mathrm{b}^2 = \mathrm{a}^2 + \mathrm{c}^2 - 2\mathrm{ac}\cos \mathrm{B} \Longrightarrow \cos \mathrm{B} = \frac{\mathrm{a}^2 + \mathrm{c}^2 - \mathrm{b}^2}{2\mathrm{ac}} = \frac{5^2 + 12^2 - 8^2}{2\mathrm{x}5\mathrm{x}12} = \frac{105}{120} = 0.8750$$

$$\mathrm{m}(< B) = \cos^{-1} 0.8750 \approx 29^0$$

$$\mathrm{c}^2 = \mathrm{a}^2 + \mathrm{b}^2 - 2\mathrm{ab}\cos \mathrm{C} \Longrightarrow \cos \mathrm{C} = \frac{\mathrm{a}^2 + \mathrm{b}^2 - \mathrm{c}^2}{2\mathrm{ab}} = \frac{5^2 + 8^2 - 12^2}{2\mathrm{x}5\mathrm{x}8} = \frac{-55}{80} = -0.6875$$

$$\mathrm{m}(< \mathrm{C_R}) = \cos^{-1} 0.6875 \approx 47^0$$

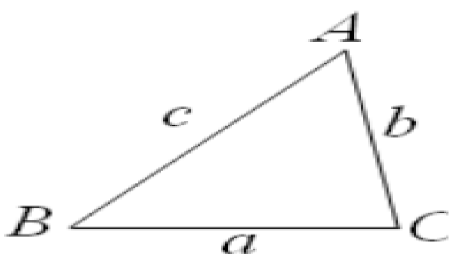
$$\mathrm{m}(< B) = 180^0 - \mathrm{m}(< \mathrm{C_R}) = 180^0 - 47^0 = 133^0$$

Activity 3.20

Given triangle ABC with $b = 4, c = 2 \; and \; m(< A) = 30^0$, find the measure of angles B and angles C of a triangle and length of BC = a.

Area law

The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle.



That is

$$\text{Area of triangle ABC} = \frac{1}{2}\text{bc}\sin\text{A}$$

$$= \frac{1}{2}\text{ac}\sin\text{B}$$

$$= \frac{1}{2}\text{ab}\sin\text{C}$$

Example; Find the area of a triangular lot ABC having lengths $\text{AB} = \text{c} = 90$ meters and $\text{BC} = \text{a} = 52$Meter and an included angle $\text{m}(< B) = 102^0$

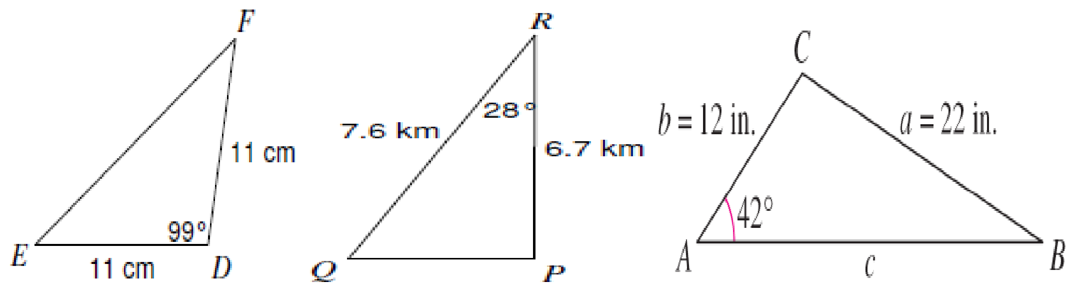
Solution;

$$\text{Area of atriangular lot ABC} = \frac{1}{2}\text{ac}\sin\text{B} = \frac{1}{2}\text{x52mx90m}\sin 102^0 = 2340\sin(180^0 - 102^0)$$

$$= 2340\sin 78^0 = 2340(0.9781) = 2288.8 \text{ square meter}$$

Activity 3.21

Find the area of oblique triangles given below;



Chapter Four
4. Coordinate Geometry

Introduction

Coordinate system or Cartesian coordinate system is sometimes known as a rectangular system used to uniquely determine a point in two or three dimensional space, by its distance from the origin of the coordinate system. It gained its name from a French mathematician and philosopher René Descartes (1596-1650). Coordinate geometry is one of the most important and exciting ideas of mathematics. In particular it is central to the mathematics students meet at school. It provides a connection between algebra and geometry through graphs of lines and curves. This enables geometric problems to be solved algebraically and provides geometric insights into algebra. Thus the simplest, most useful and most often meet application of coordinate geometry is to solve geometrical problems.

Coordinate geometry can be used to prove results in Euclidean Geometry. An important aspect of doing this is placing objects on the Cartesian plane in a way that minimizes calculations.

Coordinate geometry leads into many other topics in school mathematics. The techniques of coordinate geometry are used in calculus, functions, statistics and many other important areas.

There were three facts of the development of coordinate geometry.

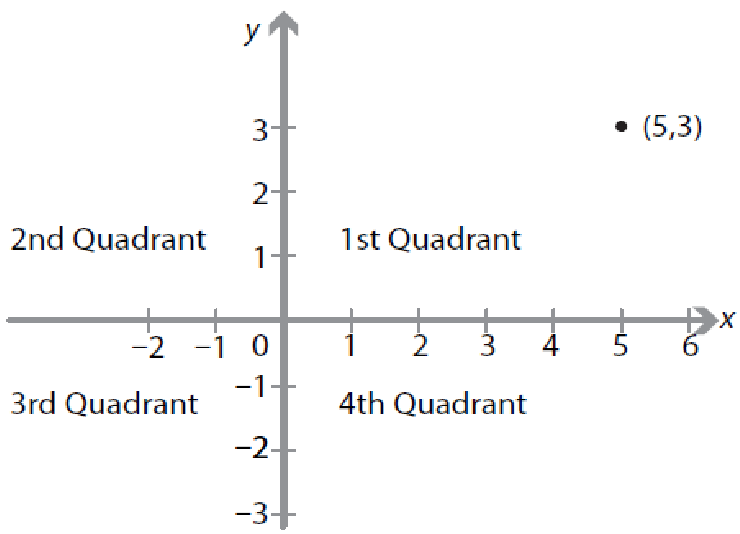
- The invention of a system of coordinates
- The recognition of the correspondence between geometry and algebra
- The graphic representation of relations and functions

In this chapter, we shall consider only the idea of Distance between points, division of line segments, distance between a point and a line, distance between two lines and general equation of a straight line and a circle.

4.1. Distance Formula

The number plane (Cartesian plane) is divided into four quadrants by two perpendicular axes called the *x*-axis (horizontal line) and the *y*-axis (vertical line). These axes intersect at a point called the **origin**. The position of any point in the plane can be uniquely represented by an ordered pair of numbers (*x*, *y*). These ordered pairs are called the coordinates of the point. Where; x is called the x-coordinate and y is called the y-coordinate.

For the point (5, 3), 5 is the *x*-coordinate and 3 is the *y*-coordinate, sometimes called the first and second coordinates. When developing trigonometry, the four quadrants are usually called the first, second, third and fourth quadrants as shown in the following diagram.



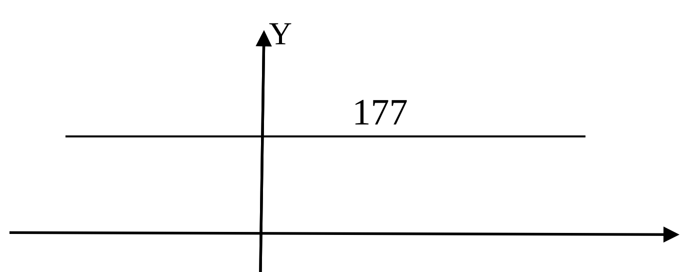
Note. Once the coordinates of two points are known the distance between the two points and midpoint of the interval joining the points can be found.

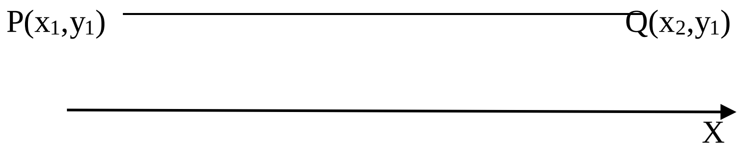
4.1.1, Distance between Two Points

Distances are always positive, or zero if the points coincide. The distance from A to Bis the same as the distance from B to A

We first find the distance between two points that are either vertically or horizontally aligned.

- When point P and Q are on a line parallel to the x-axis (horizontal line) as seen in the figure below;

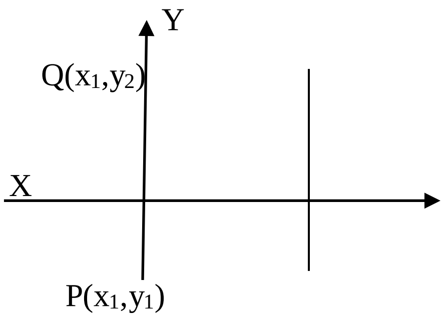




The distance between P and Q equals the length of the segment connecting the points, hence

PQ = $|x_2 - x_1|$

- When P and Q are on a line parallel to the y-axis (vertical line) as in the figure below;



The distance between P and Q equals the length of the segment connecting the points,

Hence,

PQ = $|y_2 - y_1|$

In the above two cases we have tried to see the distance between two points where the points are lie on the vertical line (parallel to the y-axis) or the horizontal line (parallel to the x-axis). Now we are going to see the case where the points are lie on a line neither parallel to the x-axis nor parallel to the y-axis.

In order to derive the formula for the distance between two points in the plane, we consider two points A(a,b) and B(c,d). We can construct a right-angled triangle ABC, as shown in the following diagram, where the point Chas coordinates (a,d).

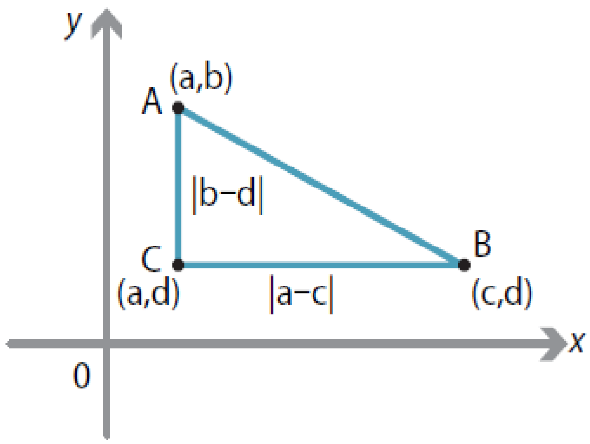


Fig 1.3

Now using Pythagoras theorem the distance between A and B is denoted by AB and we have

$$\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{CB}^2$$

$$\mathrm{AB}^2 = |\mathrm{b}-\mathrm{d}|^2 + |\mathrm{a}-\mathrm{c}|^2$$

$$= (\mathrm{a}-\mathrm{c})^2 + (\mathrm{b}-\mathrm{d})^2$$

So

$$AB = \sqrt{(a-c)^2 + (b-d)^2}$$

This formula is called **Distance formula**.

Example; Find the distance between the points

a. A(1, 2) and B(4, 2)

b. A(1, –2) and B(1, 3)

c. A (3,8) and B(11,-7).

Solution;

a. points A and B are lie on a horizontal line y = 2 which is parallel to the x – axis, so

AB = $|x_2 - x_1| = |4-1| = |3| = 3.$

b. points A and B are lie on a vertical line x = 1 which is parallel to the y –axis, so

AB = $|y_2 - y_1|= |3-(-2)|= |3+2| = |5| = 5.$

c. Points A and B are lie one a line neither parallel to the x-axis nor parallel to the y-axis, so

$$AB = \sqrt{(a-c)^2 + (b-d)^2} = \sqrt{(11-3)^2 + (-7-8)^2} = \sqrt{(8)^2 + (15)^2}$$

$$= \sqrt{64 + 225} = \sqrt{269}$$

Activity 4.1

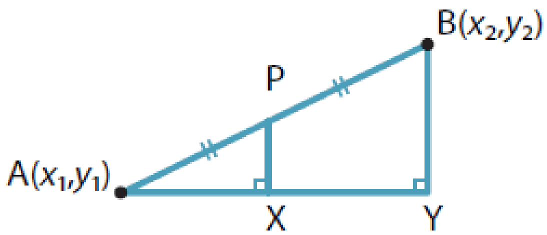
1. Find the length of the line segment whose endpoints are (6,-2) and (9,1).
2. Find the distance between the points (4,-8) and (7,-10).

4.1.2 Dividing the Line Segment in a Given Ratio.

i. The Midpoint of a Line Segment

Recall that the midpoint of a segment is the point on the segment that is equidistant from the two endpoints.

The midpoint of an interval AB is the point that divides AB in the ratio 1 : 1.



Assume that the point A has coordinates (x_1, y_1) and the point B has coordinates (x_2, y_2).

It is easy to see, using either congruence or similarity that the midpoint P of AB is

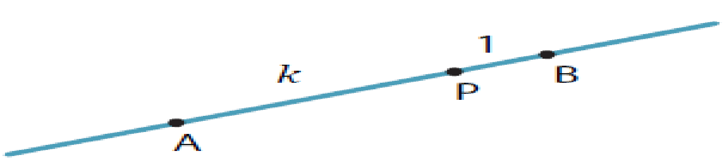
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

ii. Division of Line Segment in a Given Ratio.

We now generalize the idea of a midpoint to that of a point that divides the interval AB in the ratio k : 1.

Suppose $k > 0$ is a real number and let P be a point on a line interval AB. Then P divides AB in the ratio k : 1 means

$$\frac{AP}{PB} = \frac{k}{1} = k$$



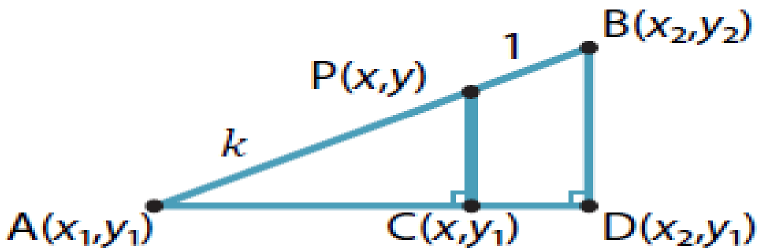
Theorem 4.1

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane and let P(x, y) be the point that divides the interval AB in the ratio k : 1, where $k > 0$. Then

$$x = \frac{x_1 + kx_2}{1+k} and y = \frac{y_1 + ky_2}{1+k}$$

Proof

If $x_1 = x_2$, then it is clear that x is given by the formula above. So we can assume that $x_1 \neq x_2$. Consider the points $C(x, y_1)$ and $D(x_2, y_1)$, as shown in the following diagram.



$\Delta ACP \sim \Delta ADB$ by AA similarity theorem, then

$$\frac{AP}{AB} = \frac{AC}{AD} = \frac{CP}{DB}$$

Now

i. $\frac{AP}{AB} = \frac{AC}{AD} \Longrightarrow \frac{k}{k+1} = \frac{x - x_1}{x_2 - x_1}$

$$\Longrightarrow k(x_2 - x_1) = (k+1)(x - x_1)$$

$$\Longrightarrow kx_2 - kx_1 = kx - kx_1 + x - x_1$$

$$\Longrightarrow kx_2 + x_1 = kx + x$$

$$\Longrightarrow kx_2 + x_1 = (k+1)x$$

$$x = \frac{kx_2 + x_1}{k+1}$$

Similarly;

ii. $\frac{AP}{AB} = \frac{CP}{DB} \Longrightarrow \frac{k}{k+1} = \frac{y - y_1}{y_2 - y_1}$

$$\Longrightarrow k(y_2 - y_1) = (k+1)(y - y_1)$$

$$\Longrightarrow k\mathrm{y}_2 - \mathrm{ky}_1 = \mathrm{k}y - k\mathrm{y}_1 + \mathrm{y} - \mathrm{y}_1$$

$$\Longrightarrow k\mathrm{y}_2 + \mathrm{y}_1 = \mathrm{k}y + \mathrm{y}$$

$$\Longrightarrow k\mathrm{y}_2 + \mathrm{y}_1 = (\mathrm{k}+1)\mathrm{y}$$

$$\mathrm{y} = \frac{k\mathrm{y}_2 + \mathrm{y}_1}{k+1}$$

In general suppose $\mathrm{A}(\mathrm{x}_1, \mathrm{y}_1)$ and $\mathrm{B}(\mathrm{x}_2, \mathrm{y}_2)$ are two points in the plane and let P(x, y) is a point that divides the line segment ABin the ratio m : n, where m $>$ 0 and n$>$ 0. Then

$$\mathrm{x} = \frac{m\mathrm{x}_2+\mathrm{nx}_1}{m+n} \text{and} \mathrm{y} = \frac{m\mathrm{y}_2+\mathrm{ny}_1}{m+n}$$

Proof; Exercise

Example

Let point A is (-3, 5) and B is (5,-10). Find

a. The distance AB
b. The midpoint P of AB
c. The point Q which divides AB in the ratio 2: 5.

Solution;

a. point A with order pair (a, b) = (-3, 5) and point B with order pair (c, d) = (5,-10) are given, then

$$AB = \sqrt{(a-c)^2 + (b-d)^2} = \sqrt{(5-(-3))^2 + (-10-5)^2} = \sqrt{(8)^2 + (-15)^2} = \sqrt{269}$$

b. let P is a point with order pair (x, y), then

$$x = \frac{x_1 + x_2}{2} = \frac{-3+5}{2} = \frac{2}{2} = 1$$

$$y = \frac{5 + (-10)}{2} = \frac{-5}{2} = -2.5$$

c. Let Q is a point with order pair (x, y), that divides line segment AB in the ratio m:n = 2:5.

Then, $\mathrm{x} = \frac{m\mathrm{x}_2+\mathrm{nx}_1}{m+n} = \frac{2(5)+5(-3)}{2+5} = \frac{10-15}{7} = \frac{-5}{7}$

$$\mathrm{y} = \frac{m\mathrm{y}_2 + \mathrm{ny}_1}{m+n} = \frac{2(-10)+5(5)}{2+5} = \frac{-20+25}{7} = \frac{5}{7}$$

Activity 4.2

1. Find the midpoint of the line segment with the given endpoints.

   a. (2, 4), (1, –3)   b. (–4, 4), (–2, 2)

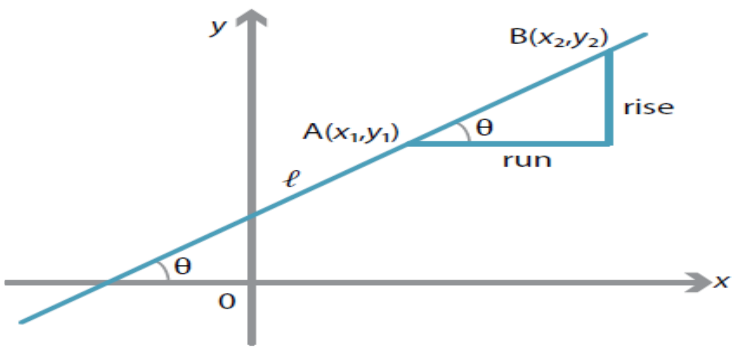
2. Find the other endpoint of the line segment with the given endpoint and midpoint.

   a. Endpoint: (5, 2), midpoint: (-10, -2)   b. Endpoint: (9, −10), midpoint: (4, 8)

3. What is the coordinate of the center of a circle if the endpoints of its diameter are A(8, −4) and B(−3, 2)?

Slopes and the Angle of Inclination

The slope is a measure of the steepness of line. Suppose *l* is a line in the number plane not parallel to the *y*-axis or the *x*-axis.



Let θ be the angle between *l* and the positive *x*-axis in the counterclockwise direction, where $0^0 \leq \theta \leq 90^0$ or $90^0 \leq \theta \leq 180^0$. then θ is called **the inclination of the line *l*.**

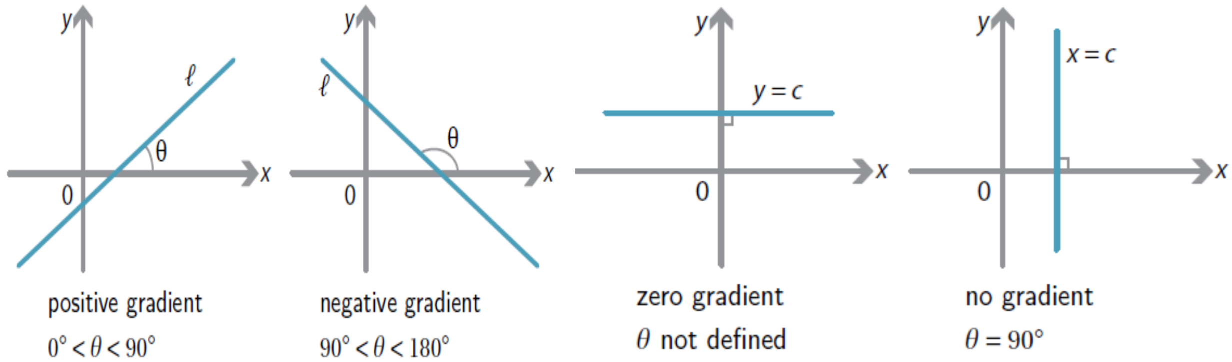
Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on l . Then, the gradient measures or the steepness or slope of the interval AB is,

$$m = \frac{VerticalRaise}{HorizontalRun} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

Provided that $x_1 \neq x_2$

So $\tan\theta$ is the slope (gradient) of the interval AB. Thus the slope (gradient) of any interval on the line is constant. Thus we may sensibly define the slope of *l* is $\tan\theta$.

$i.e$, The slope (gradient) of a line is defined to be the slope (gradient) of any interval within the line.



When the gradient or slope is 1, the line makes a 45^0 angle with either axes. If the gradient is 0, the line is parallel to the *x* axis. If the line is parallel to the *y*-axis; the gradient is not defined (the line has no gradient) because $\theta = 90^0$ and tan90 is undefined.

Activity 4.3

1. Find the gradients of the lines joining the following pairs of points. Also find the angle θbetween each line and the positive *x*-axis.

   a. (1, 2) and (7,-4) b. (2, 3) and (2,7)

2. A line passes through the point (5, 7) and has gradient 23. Find the *x*-coordinate of a point on the line when *y* = 13.
3. The line joining (2, -5) to (4, *a*) has gradient -1. Find *a*.

4.2, Equation of A Straight Line

Intercepts; All lines, except those parallel to the *x*-axis or the *y*-axis, meet both coordinate axes. Suppose that a line *l* passes through (*a*,0) and (0,*b*). Then *a* is the ***x*-intercept** and *b* is the ***y*-intercept** of *l*. The intercepts *a* and *b* can be positive, negative or zero. If the line is parallel to the x-axis and pass through the point (a, 0), then it must cross the x-axis at (a,0). If the line is parallel to the y-axis and pass through the point (0, b), then it must cross the y-axis at (0, b).

Every straight line can be represented algebraically in the form $y = mx + c$,where **m** represents the gradient of a line (its slope, steepness),**c** represents the y-intercept (a point where the line crosses the y axis)

Furthermore, there are several ways in which you can describe equation of a straight line algebraically

4.2.1, Equation of a Line Parallel to the Coordinate Axes

i. Vertical Lines

In a vertical line all points have the same *x*-coordinate, but the *y*-coordinate can take any value. In general, the equation of the vertical line through P(a, b) is x = a.

Example; The equation of the vertical line through the point (3, 0) is $x = 3$. The *x*- intercept is 3. All the points on this line have *x*-coordinate 3.

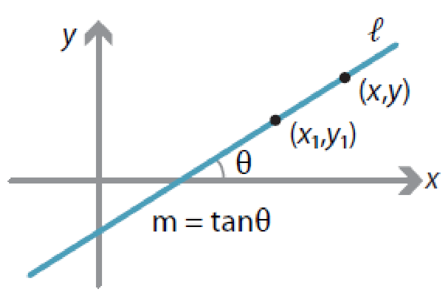
ii. Horizontal Line

A horizontal line has gradient 0. In a horizontal line all points have the same y-coordinate, but the x-coordinate can take any value. In general, the equation of the horizontal line through P(a, b) is y = b.

Example; the equation of the horizontal line through the point(0, 7) is $y = 7$. The y-intercept is7. All the points on this line have y-coordinate 7.

4.2.2, The Point-Slope form Equation of a Line

Consider the line *l* which passes through the point (x_1, y_1) and has gradient (slope)*m*.



Let P(x, y) be any point on *l*, except for (x_1, y_1). Then

$$m = \frac{y - y_1}{x - x_1}$$

and so

$$y - y_1 = m(x - x_1)$$

This is the equation of the straight line *l* with gradient **m** passing through the point (x_1, y_1) and the equation is called the **point–slope form** of equation of a line *l*.

Example; Find the equation of the line through (3,4) with slope 5.

Solution; let *l* be a line through the point (x_1, y_1)=(3, 4) with slope m= 5. Thus

$$y - y_1 = m(x - x_1) \Rightarrow y - 4 = 5(x - 3)$$

$$\Longrightarrow y = 5x - 15 + 4$$

$\Longrightarrow y = 5x - 11$ is equation of a line *l*

4.2.3. The slope-intercept form of Equation of a Line.

Suppose that $(x_1, y_1) = (0,c)$ which is the y-intercept of the line with slope m. Then the equation is

$$y - c = m(x - 0)$$

or, equivalently,

$$y = mx + c$$

Where; **c** is y-intercept and **m** is slope of the line.This is often called the **slope–intercept form** of equation of a line.

Example; the gradient of a line is –6 and the *y*-intercept is 2. Find the equation of the line

Solution; let *l* is a line with slope -6 and y-intercept 2. Then we use **slope–intercept form** of equation of a line.

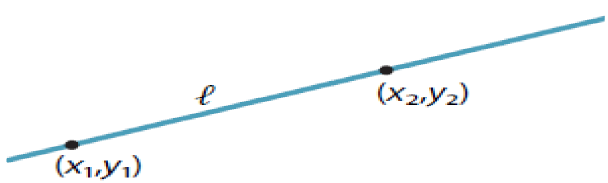
$$y = mx + c$$

$$\Longrightarrow y = -6x + 2$$

is equation of the line *l*.

4.2.4 The Two point form Equation of a Line

To find the equation of the line through two given points (x_1, y_1) and (x_2, y_2), first find the gradient (slope);



$$m = \frac{y_2 - y_1}{x_2 - x_1}, \qquad provided\ x_1 \neq x_2$$

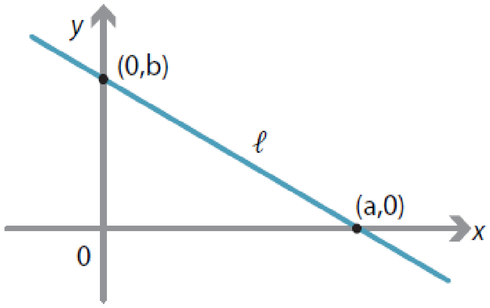
and then use the point–slope form

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This equation is called the **two point form** of equation of a line *l*.

A special case is the line through (a, 0) and (0, b), where a, b ≠ 0.



In this case, the gradient (slope) is

$$m = \frac{b-0}{0-a}$$

Thus the equation of the line is

$$y - 0 = \frac{-b}{a}(x-a)$$

$$ay = -b(x-a)$$

$$ay + bx = ab$$

This is called the **intercept form** of the equation of a line.

Activity 4.4

1. Find the equation of the line through (3,4) and (-2,-3).
2. The lines y = 4x – 7 and 2x + 3y – 21 = 0 intersect at point A. The point B has coordinates (-2, 8). Find the equation of the lines that passes through points A and B.

4.2.5 General Form of Equation of a Straight Line

One of the axioms of Euclidean geometry is that two points determine a line. In other words, there is a unique line through any two fixed points. This idea translates to coordinate geometry and, as we shall see, all points on the line through two points satisfy an equation of the form ax + by + c = 0 with a and not both 0. Conversely, any 'linear equation' $ax + by + c = 0$ is the equation of a (straight) line where a, b and c are real number. This is called the **general form** of the equation of a line.

Example; Find the general form of equation of a line *l* on the point (3, -5) and meet the line with equation 7x-3y =2 at x = 2.

Solution; 7x-3y = 2, $\Longrightarrow y = \frac{7}{3}x - \frac{2}{3} at x = 2\ the value of y is\ 4$

$i.e\ the\ points\ (3,-5) and\ (2,4)\ must\ lie\ on\ the\ line\ l$, (x_1, y_1) = (3, -5)and (x_2, y_2) = (2, 4),

This implies the equation of the line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - (-5) = \frac{4 - (-5)}{2 - 3}(x - 3) = \frac{9}{-1}(x - 3)$$

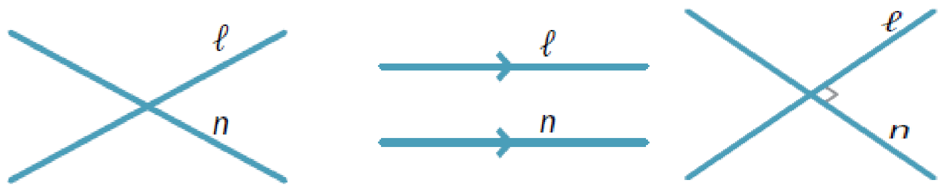
$$y = -9x + 27 - 5$$

$$y = -9x + 22$$

Thus ,$y + 9x - 22 = 0$ is the general form of equation of a line *l*

4.3 Parallel, Intersecting and Perpendicular lines

The axioms for Euclidean geometry include: Two lines meet at a point or are parallel. Among those pairs of lines which meet, some are perpendicular.



The most important thing you need to know about parallel and perpendicular lines is that the relationship between parallel lines is a relationship between the slope of the lines, and the same goes for perpendicular lines.

4.3.1 Parallel Lines

Two lines are **parallel lines,** if they lie in the same plane and do not intersect. On the above figures the middle ones, lines *l* and *n* are parallel lines. You can write this as *l* // *n*.

If two lines $l_1$and $l_2$are parallel then corresponding angles with the reference of the horizontal line are equal. Conversely, if corresponding angles with the reference of the horizontal line are equal then the lines are parallel.

Clearly, two horizontal lines are parallel. Also, any two vertical lines are parallel. If lines *l* and *n* are not parallel, then their point of intersection can be found by solving the equations of the two lines simultaneously.

Theorem 4.2

Two lines are parallel if they have the same slope and conversely, two lines with the same slope are parallel.

Proof; Exercise

This property is fairly easy to understand why it is true. Remember that the slope of a line represents the steepness of the line. So, if two lines are parallel, they would have to have the same steepness, otherwise they would eventually intersect, making them no longer parallel by definition.

Example; Show that the line passing through the points A(6, 4) and B(7, 11) is parallel to the line passing through P(0, 0) and Q(2, 14).

Solution: let l_1 is a line passing through the points A(6, 4) and B(7, 11), then the slope is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 4}{7 - 6} = 7$$

And letl_2 is a line pass through the points P(0, 0) and Q(2, 14), then the slope is

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 0}{2 - 0} = \frac{14}{2} = 7$$

Both $l_1 and l_2$ are the same slope, and then they have parallel to each other,

If two non-vertical lines are parallel then they have the same slopes. Conversely if two non-vertical lines have the same slopes then they are parallel.

4.3.2 Perpendicular Lines

Two lines are **perpendicular lines**, if they intersect to form a right angle. Lines *s* and *t* are perpendicular lines, you can write this as $s \perp t$. All vertical lines x = a are perpendicular to all horizontal lines y = b.

Two non- vertical lines are perpendicular if and only if the product of their slopes is -1. Conversely if the product of the slopes of two lines is –1 then they are perpendicular.

Note:let line $l_1$hasslope $\mathrm{m}_1$and cross the y − axis at y = c, line l_2 has slope $\mathrm{m}_2$and cross the y-axis at y = d, then

- ✓ If $m_1 = m_2 and c \neq d$ then $l_1 // l_2$.
- ✓ If $m_1 = m_2 and c = d$ then l_1 coinsideswithl_2 (they are identical).
- ✓ If $m_1 = -\frac{1}{m_2}$ then $l_1 \perp l_2$
- ✓ If $m_1 \neq m_2 then l_1 and l_2$ meet at a point.

Activity 4.5

1. Find the equation of the line *l* through (1,3) perpendicular to the line 2*x* + 3*y* = 12. Find the equation of the line through (4,5) parallel to *l* .
2. Find the equation of the line which passes through the point (1, 3) and is perpendicular to the line whose equation is *y* = 2*x* + 1.
3. Determine if the two lines are parallel.
   a. The line passing through the points (-2, 1) and (4, 3)
      The line passing through the points (3, -2) and (5, -1)

b. 5x +6y =1 and 2-5x-3y =10

4. Find equation of the line perpendicular to $3x + 4y = 4$, passing through the point (-4,-2).

5. For each pair of lines, determine whether they are parallel, identical or meet. If they meet, find the point of intersection and whether they are perpendicular.

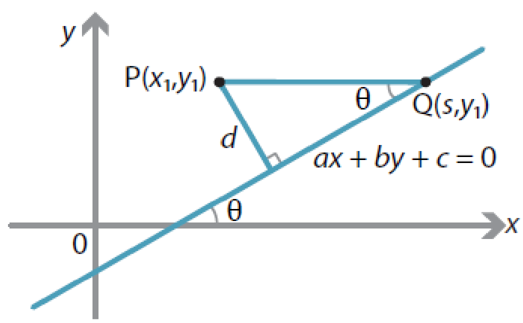
   a. y = 2x -5 and y = 5x -5

   b. 2y = 8x -1 and 4y -16x +2 = 0

4.4 Perpendicular Distances

4.4.1 Distance between a Point and a Line

Given a line *l* with equation ax + by + c = 0$a \neq 0, b \neq 0$ and a point P(x_1, y_1)not on the line, then the distance **d** of P from *l* is;



$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Example; how far is the point (3,-2) from the line 2x +3y -2 = 0?

Solution;$x_1 = 3,\ y_1 = -2,$ a = 2, b = 3, and c = -5, then

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \frac{|2x3 + 3x(-2) + (-5)|}{\sqrt{2^2 + 3^2}} = \frac{|6 - 6 - 5|}{\sqrt{4 + 9}} = \frac{|-5|}{\sqrt{13}} = \frac{5\sqrt{13}}{13}$$

4.4.2 Distance between Two Lines;

Given any two lines, they could have the following three properties;

- ✓ They may cross each other
- ✓ They may coincide or
- ✓ They may not cross each other.

The distance between two intersecting lines or coincides to each other is always zero. To deal about distance between two lines, consider the lines are parallel otherwise the distance between these lines is always zero.

The distance between two parallel lines is the same as the distance between the point lie on one of the straight line to other straight line.

Steps to find distance between two parallel lines;

i. Take a point from one of the given lines

ii. Find the distance from the point taken in step one and the other lines

iii. The distance obtained in step two is the same as the distance between the two lines

Example; find the distance between the following pair of lines;

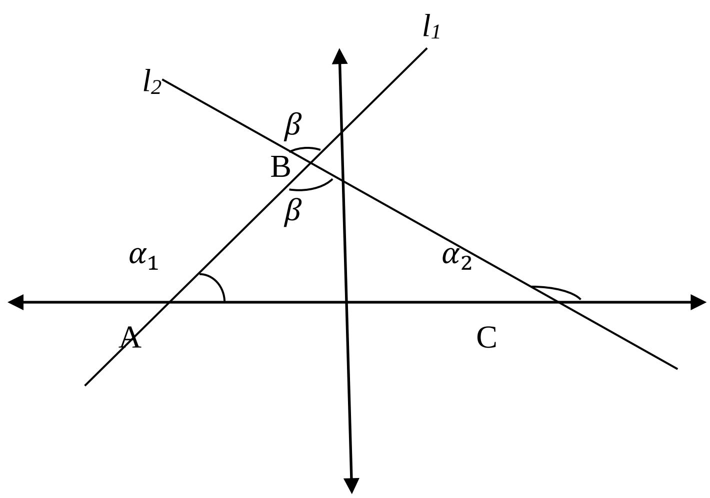
a. $5x + 12y = 26 \text{ and } 12y = -5x + 20$

b. $2x - 7y = 24 \text{ and } 3x + 2y = -6$

c. $14x + 30y = 42 \text{ and } 7x + 15y - 21 = 0$

4.4.3 Angle between Two Intersecting Lines

Let $0 \le \beta \le \pi$ is an angle between two intersecting lines l_1 and l_2 have slope m_1 and m_2 respectively; measured in counterclockwise direction, then $\pi - \beta$ is also an angle between l_1 and l_2. Assume that α_1 is an angle between the line l_1 and the positive x-axis and α_2 is an angle between the line l_2 and the positive x-axis, both measured in counterclockwise direction. Therefore $m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$.



In ΔABC the exterior angle is equal to the sum of the two opposite interior angles

$i.e;\ \alpha_1 + \beta = \alpha_2 \Longrightarrow \beta = \alpha_2 - \alpha_1$

Hence

$$\tan\beta = \tan(\alpha_2 - \alpha_1)$$

$= \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2}$; if $m_1 m_2 \neq -1$

$$\boldsymbol{\beta = \arctan\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)}$$

Example; Find the angle between the lines $y - 2x - 1 = 0$ and $y + 3x - 2 = 0$.

Solution; Let β is an angle between these two lines; slope of a line with equation with equation $y - 2x - 1 = 0$ is $m_1 = 2$ and Slope of a line with equation with equation $y + 3x - 2 = 0$ is $m_2 = -3$, then

$$\frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-3 - 2}{1 + 2(-3)} = \frac{-5}{1 - 6} = \frac{-5}{-5} = 1$$

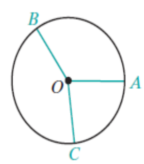
$$\beta = \arctan\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right) = \arctan(1) = 45^0 = \frac{\pi}{4}$$

Then, the angle between the lines $y - 2x - 1 = 0$ and $y + 3x - 2 = 0$ $is \beta = \frac{\pi}{4}$.

4.5 Equation of a Circle.

Definition of a Circle

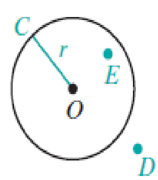
Definition:-A **circle** is the set of all points in a plane that are equidistant from a fixed point on the plane called the **center** of the circle.



If the center of a circle is point O, the circle is called **circle O**. A **radius** of a circle (plural, *radii*) is a line segment from the center of the circle to any point of the circle. If A, B, and C are points of circle O, then$\overline{OA}$,$\overline{OB}$ and $\overline{OC}$are radii of the circle. Since the definition of a circle states that all points of the circle are equidistant from its center O, $OA = OB = OC$, This implies that all radii of a circle are congruent (equal in length).

Circle separates a plane into three sets of points. If we let the length of the radius of circle O is r, then:

- Point *C* is on the circle if $OC = r$.
- Point *D* is outside the circle if $OD > r$.
- Point *E* is inside the circle if $OE < r$.

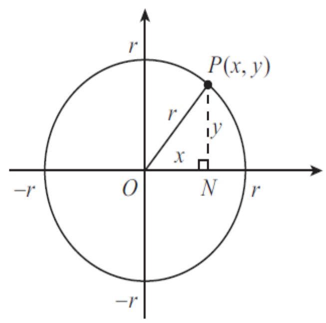


The **interior of a circle** is the set of all points whose distance from the center of the circle is less than the length of the radius of the circle.

The **exterior of a circle** is the set of all points whose distance from the center of the circle is greater than the length of the radius of the circle.

i. Equation of a Circle with Centered at the Origin.

The simplest case is that of a circle whose centre is at the origin. If we take any point P(x, y) on the circle O, then OP = r is the radius of the circle. From the figure below, OP is the hypotenuse of the right-angled triangle OPN, formed when we drop a perpendicular from P to the x-axis. In the right-angled triangle OPN, ON = x and NP = y.



Thus, using the Pythagoras theorem,

$$x^2 + y^2 = r^2$$

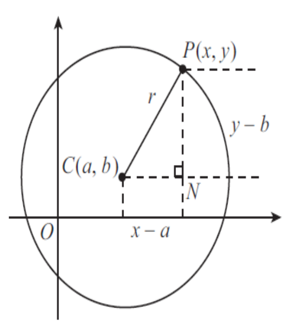
This is the equation of a circle with radius r and centre at the origin O(0, 0).

Thus,

The equation of a circle of radius r and centre at the origin is $$x^2 + y^2 = r^2$$

iii. The General Equation of a Circle

The general form of equation of a circle C of radius r, centered at the point C(a, b)$\neq$ O(0, 0). If we take any point P(x, y) on the circle C,



We shall take a horizontal line through the centre C and drop a perpendicular from P to meet this horizontal line at $N(x, b)$. Then again we have a right-angled triangle CPN, where $CP = r$ is the hypotenuse, and where we have $CN = x - a$ and $PN = y - b$. Thus using Pythagoras theorem we have;

$$CN^2 + PN^2 = CP^2$$

$$\Longrightarrow (x - a)^2 + (y - b)^2 = r^2$$

The standard form of general equation of a circle with center at C (a, b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2 = k$$

Depending on the value of k, the following situations occur

- k >0The equation represent a circle with center (a, b) and radius $r = \sqrt{k}$
- k = 0The only solution of the equation is x = a, y = b, so the equation represent a single point (a, b).
    - k<0The equation has no real solutions and consequently doesn't represent equation of a circle.

From the above standard form equation of a circle expanding the brackets gives

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

Bring r^2 to the left hand side and rearrange we get

$$x^2 - 2ax + y^2 - 2by + a^2 + b^2 - r^2 = 0$$

Let $d = a^2 + b^2 - r^2$; then

$$x^2 - 2ax + y^2 - 2by + d = 0$$

is the general form of equation of a circle C, with center C(a, b) and radius

$$r = \sqrt{a^2 + b^2 - d}$$

Thus;

The general equation of a circle is

$$x^2 - 2ax + y^2 - 2by + d = 0$$

Where the center is C (a, b) and radius $r = \sqrt{a^2 + b^2 - d}$

Exa

2. Find the equation of a circle passing through the point (5, 12) with centre at the origin.

   Solution; given P(x, y) = (5, 12), but the equation of a circle of radius r and centre at the origin is

$$x^2 + y^2 = r^2$$

$$r^2 = x^2 + y^2 = 5^2 + 12^2 = 25 + 144 = 169$$

Thus the equation of a circle is

$$x^2 + y^2 = r^2 = 169$$

3. What is the radius and the centre of a circle with equation $(x - 2)^2 + (y - 5)^2 = 18$?

   Answer; the radius is $3\sqrt{2}$ and the center is (2, 5).

4. Find the centre and radius of the circle represented by the equation

$$x^2 + y^2 + 10x + 6y - 2 = 0$$

Solution; rewrite the equation $as\ \ x^2 - 2(-5)x + y^2 - 2(-3)y - 2 = 0$

This implies that a = -5, b = -3 and d = -2 and then the center is C(a, b) = (-5, -3) and

Radius$r = \sqrt{a^2 + b^2 - d} = \sqrt{(-5)^2 + (-3)^2 - (-2)} = \sqrt{25 + 9 + 2} = \sqrt{36} = 6$

Activity 4.6

1. Find the center and radius of the circle with equation

   a. $x^2 + y^2 - 8x + 2y + 8 = 0$ d. $2x^2 + 2y^2 - 8x - 7y = 0$.

   b. $(x + 1)^2 + (y + 3)^2 = 5$ e. $x^2 + y^2 - 4x - 6y + 13 = 0$

   c. $x^2 + (y + 2)^2 = 1$ f. $x^2 + y^2 + 8x + 8 = 0$

Exercise 4.4

1. The distance between the points (1, a) and (4, 8) is 5. Find the possible values of a.
2. Show that the distance between the points A(a, b) and B(c, d) is the same as the distance between the points P(a, d) and Q(c, b).
3. Find the midpoint of the line segment with the given endpoints.

   a. (–4, 4), (5, –1) b. (–1, –6), (–6, 5)

4. Find the other endpoint of the line segment with the given endpoint and midpoint.

   a. Endpoint: (-1, 9), midpoint: (-9, -10) b. Endpoint: (2, 5), midpoint: (5, 1)

5. One endpoint of a line segment is (8, −1). The point (5, −2) is one-third of the way from that endpoint to the other endpoint. Find the other endpoint.
6. A square has vertices O(0, 0), A(6, 0), B(6, 6) and C(0, 6).

   a. Find the midpoint of the diagonals OB and CA.

   b. Find the length of a diagonal of the square

7. Find the gradients of the lines joining the following pairs of points. Also find the angle θ between each line and the positive x-axis.

   a. (2, 3) and (5,3) b. (1, 2) and (7,8)

8. Find the gradients of the lines joining the following pairs of points:

   a. $(ap^2, 2ap), (aq^2, 2aq)$

   b. $(a\cos\theta, b\sin\theta), (a\cos\beta, b\sin\beta)$.

9. Find the equation of the line with x-intercept 3 and y-intercept -5.

10. An equilateral triangle OBC has coordinates O(0, 0), B(a, 0) and C(c, d).

    a. Find c and d in terms of a by using the fact that OB = BC = CO.

    b. Find the equation of the lines that contains the intervals OB, BC and CO.

11. The line $y = 2x - 4$ meets the x axis at point A. Find the equation of the line with gradient 2/3 that passes through point A.

12. Find the equation of the line passing through the origin and perpendicular to the line whose slope is 1.

13. Find the equation of the line passing through the (3,-1) and parallel to the line with equation $2x + 4y = 1$.

14. Show that the line through the points A(6, 0) and B(0, 12) is perpendicular to the line through P(8, 10) and Q(4, 8).

15. Determine whether or not the two lines are parallel.

    a. The line passing through the points (5,-2) and (6, 2)

       The line passing through the points (4, -3) and (8, 4)

    b. $ax + by = c$ and $ax + by = 0$, $a, b, c \neq 0$

16. For each pair of lines, determine whether they are parallel, identical or meet. If they meet, find the point of intersection and whether they are perpendicular.

    a. $4y - 3x - 18 = 0$ and $3y + 4x - 1 = 0$

    b. $2y = 6x + 12$ and $y - 3x = 7$

17. Write the equation of a circle with**;**

   a. Center: (13, −13)
      Radius: 4

   b. Center: (−13, −16)
      Point on Circle: (−10, −16)

   c. Ends of a diameter: (18, −13) and (4, −3)
      center at the origin

   d. Center: (0, 13)
      Area: 25π

Summary of the unit

- The number plane (Cartesian plane) is divided into four quadrants by two perpendicular axes called the *x*-axis (horizontal line) and the *y*-axis (vertical line). The position of any point in the plane can be uniquely represented by an ordered pair of numbers (*x*, *y*). These ordered pairs are called the coordinates of the point.
- Distances are always positive, or zero if the points coincide. The distance from A to B is the same as the distance from B to A.
- The distance between point A has coordinates (x_1, y_1) and point B has coordinates (x_2, y_2) is;

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in the plane and let P(x, y) is a point that divides the line segment AB in the ratio m : n, where m > 0 and n> 0. Then

$$x = \frac{mx_2 + nx_1}{m+n} \text{ and } y = \frac{my_2 + ny_1}{m+n}$$

- The slope is a measure of the steepness of line. **The inclination of the line *l*** is the angle θ between *l* and the positive x-axis in the counterclockwise direction, where $0^0 \leq \theta \leq 90^0$ or $90^0 \leq \theta \leq 180^0$. Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on l . Then, the slope of the interval AB is,

$$\mathbf{m} = \frac{\textbf{Vertical Raise}}{\textbf{Horizontal Run}} = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}} = \mathbf{\tan\theta}$$

Provided that $x_1 \neq x_2$

- Form of equation of a straight line;
  - Equation of a Line Parallel to the Coordinate Axes
  - The Point-Slope form Equation of a Line
  - The slope-intercept form of Equation of a Line
  - The Two point form Equation of a Line
  - General Form of Equation of a Straight Line
- Two lines are **parallel lines,** if they lie in the same plane and do not intersect.
- Two lines are **perpendicular lines**, if they intersect to form a right angle.
- Let line l_1 has slope m_1 and cross the y-axis at y = c, line l_2 has slope m_2 and cross the y – axis at y = d, then;
  - If $m_1 = m_2 and c \neq d$ then $l_1 // l_2$.
  - If $m_1 = m_2 and c = d$ then $l_1 coinsides with l_2 (theyareidentical)$.
  - If $m_1 = -\frac{1}{m_2}$ then $l_1 \perp l_2$
  - If $m_1 \neq m_2 then l_1 and l_2$ meet at a point.

- Given a line l with equation ax + by + c = 0a$\neq 0, b \neq 0$ and a point P(x_1, y_1)not on the line, then the distance **d** of P from l is;

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

- The distance between two intersecting lines or coincides to each other is always zero. The distance between two parallel lines is the same as the distance between the point lie on one of the straight line to other straight line.
- Let $0 \leq \beta \leq \pi$ is an angle between two intersecting lines l_1 and $l_2$measured in counterclockwise direction have slope $m_1$and m_2 respectively. Then;

$$\beta = \arctan(\frac{m_2 - m_1}{1 + m_1 m_2})$$

- A **circle** is the set of all points in a plane that are equidistant from a fixed point on the plane called the **center** of the circle.
- Equation of a circle with radius r and centre at the origin and point p(x, y) lie on circle is

$$x^2 + y^2 = r^2$$

- The general equation of a circle is

$$x^2 - 2ax + y^2 - 2by + d = 0$$

Where the center is C (a, b) and radius $r = \sqrt{a^2 + b^2 - d}$

- The standard form of general equation of a circle with center at C (a, b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2 = k$$

Reference

1. Alemayehu Haile and YismawAlemu (Phd): "**Mathematics; An introductory course**"
2. Grade 10, 11 and Grade 12 Mathematics student text both old and new curriculum.
3. Harikishan; " **coordinate geometry of two dimensions**"
4. As. Professor David Hunt:" **coordinate geometry a guide for teachers**"
5. Asseged Atnafu: A comprehensive Guide to High School Mathematics.
6. Dan kennedy (phD) and Randall I. Charles (PhD) virgina prentice hall Mathematics, Algebra 2,
7. Dr. William Tawdross E beid – vice dean. Faculty of Education A in shams university Mathematics for first year preparatory schools
8. **H.L a'urence Ridge – Faculty of educations university of Toronto (Math scope 2)**
9. **Kenya Mathematics: Teacher's Edition Algebra: Clyde A. Dilley.** Mathematics an Introductory course, Department of Mathematics AAU, Addis Ababa.
10. Yohannes Woldetensae (PhD) - Basic Algebra with applications, Africa Beza college
11. Virgina Mathematics with an integrated approach – **Mc Dougal Little /Houghton Mifflin/**